

THE RADON TRANSFORM ON THE HEISENBERG GROUP AND THE TRANSVERSAL RADON TRANSFORM

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ABSTRACT. The notion of the Radon transform R_H on the Heisenberg group \mathbb{H}_n was introduced by R. Strichartz and inspired by D. Geller and E.M. Stein's related work. The transversal Radon transform R_T integrates functions on \mathbb{R}^m , $m \geq 2$, over hyperplanes meeting the last coordinate axis. In fact, R_H is a particular case of R_T , corresponding to $m = 2n + 1$. We obtain new boundedness results and explicit inversion formulas for both transforms on L^p functions in the full range of the parameter p . We also show that R_H and R_T are isomorphisms of the corresponding Semyanistyi-Lizorkin spaces of smooth functions. In the framework of these spaces we obtain inversion formulas, which are pointwise analogues of the corresponding formulas by R. Strichartz.

1. INTRODUCTION

Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with the multiplication law

$$(z, t) \circ (\zeta, \tau) = (z + \zeta, t + \tau - \frac{1}{2} \operatorname{Im}(z \cdot \bar{\zeta}))$$

and let

$$(1.1) \quad (R_H f)(z, t) = \int_{\mathbb{C}^n} f((z, t) \circ (\zeta, 0)) d\zeta = \int_{\mathbb{C}^n} f(\zeta, t - \frac{1}{2} \operatorname{Im}(z \cdot \bar{\zeta})) d\zeta$$

be the Radon-like transform, which was introduced by Strichartz in his remarkable paper [Str2]. He called R_H the Heisenberg Radon transform and noted that definition (1.1) stems from the previous work by Geller and Stein [GS1, GS2]. The Heisenberg Radon transform represents an interesting object from the point of view of both harmonic analysis and integral geometry. Related transforms in the more general context of

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nilpotent Lie groups and Siegel domains were studied by Felix [Fe1, Fe2], He and Liu [HL1, HL2], Peng and Zhang [PZ]; see, also [He1, He2, LH, NT].

A variety of deep results in the area of harmonic analysis on the Heisenberg group can be found in fundamental works by M.L. Agranovsky, C. A. Berenstein, D.C. Chang, M. Cowling, G.B. Folland, D. Geller, A. Korányi, D. Müller, D.H. Phong, F. Ricci, E.M. Stein, S. Thangavelu, J. Tie, and many other authors; see, e.g., [BCT, CK, KR, Kor, MRS, St2, Than], and references therein.

Owing to [Str2, p. 386, 387], in a slightly different notation we have the mixed norm estimate

$$(1.2) \quad \left[\int_{\mathbb{C}^n} \left(\int_{\mathbb{R}} |(R_H f)(z, t)|^r dt \right)^{p'/r} d\zeta \right]^{1/p'} \leq c \|f\|_p,$$

$$1 \leq p \leq p_0, \quad p_0 = 1 + 1/(2n+1), \quad r = p/(2n+1-2np), \quad 1/p + 1/p' = 1.$$

The paper [Str2] (see pp. 386, 399) also contains the inversion formulas

$$(1.3) \quad R_H^{-1} = (4\pi)^{-2n} \left(\frac{\partial}{\partial t} \right)^n R_H \left(\frac{\partial}{\partial t} \right)^n, \quad R_H^{-1} = \mathcal{F}^{-1} J \mathcal{F}_2,$$

where \mathcal{F}_2 denotes the Fourier transform in the t -variable, \mathcal{F} is the Fourier transform in all variables, and $(Jf)(x, y) = f(-2y^{-1}x, y)$.

Our research is motivated by the following questions, that stem from [Str2] and remained open in afore-mentioned publications.

Question 1. Is $R_H f$ well defined a.e. for $f \in L^p(\mathbb{H}_n)$, when $p > p_0$? If yes, then what can we say about the boundedness of R_H in this case?

Question 2. What is the substitute of (1.3) for $f \in L^p(\mathbb{H}_n)$?

Question 3. To what class of functions are formulas (1.3) applicable pointwise?

In the present article we answer these questions in a more general context of the so-called transversal Radon transform R_T , which includes R_H as a particular case. Our method differs in principle from those in the previous related works [Str2, Fe1, Fe2, HL1, HL2, PZ]. To be more specific, let us rewrite (1.1) by setting $z = u + iv, \zeta = \xi + i\eta$, where $u, v, \xi, \eta \in \mathbb{R}^n$. We obtain

$$(R_H f)(u + iv, t) = \int_{\mathbb{R}^{2n}} f(\xi, \eta, t - \frac{1}{2}(v \cdot \xi - u \cdot \eta)) d\xi d\eta.$$

Then we identify $f(\xi + i\eta, \tau)$ with a function $f(x)$ on \mathbb{R}^m , $m = 2n+1$, where

$$(1.4) \quad x \equiv (x', x_m) \in \mathbb{R}^m, \quad x' = (\xi, \eta) \in \mathbb{R}^{m-1}, \quad x_m = \tau \in \mathbb{R},$$

and set

$$(1.5) \quad a = \frac{1}{2}(-v, u) \in \mathbb{R}^{m-1}, \quad b = t \in \mathbb{R}.$$

This gives

$$(1.6) \quad (R_H f)(u + iv, t) = (R_T f)(a, b),$$

where

$$(1.7) \quad (R_T f)(a, b) = \int_{\mathbb{R}^{m-1}} f(x', a \cdot x' + b) dx'.$$

Integral (1.7) can be regarded as a Radon transform of a function f on \mathbb{R}^m associated with the hyperplane

$$(1.8) \quad h = \{x = (x', x_m) \in \mathbb{R}^m : x_m = a \cdot x' + b\},$$

which is transversal to the last variable. Following Strichartz [Str2, p. 385], we call (1.7) the *transversal Radon transform*. It resembles a special case of the affine Radon transform studied in Gelfand's school [GGG, GGV] and the parametric Radon transform in [Ehr]. To distinguish notation, we write

$$(a, b) \in \tilde{\mathbb{R}}^m = \mathbb{R}^{m-1} \times \mathbb{R}.$$

Note that m in (1.7) may be an arbitrary integer ≥ 2 . The case of m odd agrees with the Heisenberg Radon transform R_H . Thus, basic results for R_H will follow from those for R_T .

Main results. We invoke the classical Radon transform

$$(1.9) \quad (Rf)(h) = \int_h f(x) d_h(x),$$

where integration over the hyperplane h in \mathbb{R}^m is performed against the usual Lebesgue measure. Suppose that h is parameterized as

$$(1.10) \quad h = \{x \in \mathbb{R}^m : x \cdot \theta = t\}, \quad (\theta, t) \in \mathbb{P}^m = S^{m-1} \times \mathbb{R},$$

S^{m-1} being the unit sphere in \mathbb{R}^m . Then

$$(1.11) \quad (Rf)(h) \equiv (Rf)(\theta, t) = \int_{\theta^\perp} f(y + t\theta) dy.$$

The corresponding dual transform is

$$(1.12) \quad (R^* \psi)(x) = \int_{S^{m-1}} \psi(\theta, x \cdot \theta) d\theta.$$

The duality relation

$$\int_{\mathbb{P}^m} (Rf)(\theta, t) \psi(\theta, t) d\theta dt = \int_{\mathbb{R}^m} f(x) (R^* \psi)(x) dx$$

holds provided that at least one of these integrals is finite when f and ψ are replaced by $|f|$ and $|\psi|$, respectively.

Integral transforms (1.9) and (1.12), are the most familiar to the reader. They were introduced by Radon [Rad] and studied in numerous publications together with their k -plane generalizations; see, e.g., [Ber, Dea, GGG, Hel, Kat, Mar, Na, Pal, QCK, RK, Ru7, SSW] and references therein.

Unlike (1.10), that parameterizes *all* hyperplanes in \mathbb{R}^m , (1.8) excludes hyperplanes parallel to the x_m -axis. If both parameterizations are available, then the standard Calculus yields

$$(1.13) \quad (Rf)(h) = \sqrt{1 + |a|^2} (R_T f)(a, b).$$

Equation (1.13) means that basic properties of R_H and R_T can be derived from known facts for R . However, as indicated by Strichartz [Str2, p. 385], there are essential distinctions between R_T and R . This remark inspires us to give direct proofs, that might be instructive.

Theorem 1.1. *Let $\operatorname{Re} \alpha > 0$,*

$$\lambda = \frac{\pi^{(m-1)/2} \Gamma(\alpha/2)}{\Gamma((\alpha+m-1)/2)}.$$

The following equalities hold:

$$(1.14) \quad \int_{\tilde{\mathbb{R}}^m} (R_T f)(a, b) \frac{|b|^{\alpha-1} da db}{(1 + |a|^2)^{(\alpha+m-1)/2}} = \lambda \int_{\mathbb{R}^m} f(x) |x|^{\alpha-1} dx,$$

$$(1.15) \quad \int_{\tilde{\mathbb{R}}^m} (R_T f)(a, b) \frac{|b|^{\alpha-1} da db}{(1 + |a|^2 + b^2)^{(\alpha+m-1)/2}} = \lambda \int_{\mathbb{R}^m} f(x) \frac{|x|^{\alpha-1}}{(1 + |x|^2)^{\alpha/2}} dx,$$

provided that either side of the corresponding equality is finite when f is replaced by $|f|$.

Equalities (1.14) and (1.15) are proved in Section 2 and play a key role. As we shall see in Sections 3 and 4, (1.14) paves the way to the “right” definition of the *dual transversal Radon transform*, the corresponding analytic families of fractional integrals of the Semyanistyi type, and a variety of explicit inversion formulas for R_T . Equality (1.15) answers the question about existence of the transversal Radon transform of L^p functions and provides some boundedness results. In particular, by Hölder’s inequality, (1.15) implies the following corollary (cf. [Ru7, Corollary 2.4]) .

Corollary 1.2. *If $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$, then $(R_T f)(a, b)$ is finite for almost all $(a, b) \in \tilde{\mathbb{R}}^m$. Moreover, for any $\alpha > 1 - m/p'$,*

$1/p + 1/p' = 1$, we have

$$\int_{\mathbb{R}^m} |(R_T f)(a, b)| \frac{|b|^{\alpha-1} da db}{(1 + |a|^2 + b^2)^{(\alpha+m-1)/2}} \leq c \|f\|_{L^p(\mathbb{R}^m)},$$

where $c = c(\alpha, p, m) = \text{const} < \infty$.

Corollary 1.2 leads to the following result for the Heisenberg Radon transform (just set $m = 2n + 1$).

Theorem 1.3. *Let $p_1 = 1 + 1/2n$. If $f \in L^p(\mathbb{H}_n)$, $1 \leq p < p_1$, then $(R_H f)(z, t)$ is finite for almost all $(z, t) \in \mathbb{H}_n$. Moreover, for any $\alpha > 1 - (2n + 1)/p'$,*

$$\int_{\mathbb{H}_n} |(R_H f)(z, t)| \frac{|t|^{\alpha-1} dz dt}{(1 + |z|^2 + t^2)^{(\alpha+2n)/2}} \leq c \|f\|_{L^p(\mathbb{H}_n)}.$$

Clearly, $p_1 = 1 + 1/2n$ in Theorem 1.3 is greater than $p_0 = 1 + 1/(2n + 1)$ in (1.2). This gives the answer to Question 1. Note also that the condition $p < m/(m - 1)$ in Corollary 1.2 (resp. $p < p_1$ in Theorem 1.3) is sharp. This follows from (1.13) and the well-known fact that if $p \geq m/(m - 1)$ and

$$f(x) = (2 + |x|)^{-m/p} (\log(2 + |x|))^{-1} \in L^p,$$

then $(Rf)(h) \equiv \infty$; see, e.g., [So, Ru7].

Theorem 1.4. *For $m \geq 2$ an a priori inequality*

$$\left(\int_{\mathbb{R}^{m-1}} \left[\int_{\mathbb{R}} |(R_T f)(a, b)|^r db \right]^{q/r} da \right)^{1/q} \leq c_{p,q,r} \|f\|_p$$

holds if and only if

$$(1.16) \quad 1 \leq p < m/(m - 1), \quad q = p', \quad 1/r = 1 - m/p'.$$

This statement will be proved in Section 8. For $m = 2n + 1$, it extends Strichartz's result (1.2) to all $1 \leq p < p_1$, where the bound $p_1 = 1 + 1/2n > p_0$ is best possible.

A variety of inversion formulas for the transversal Radon transform and the Heisenberg Radon transform are presented by Theorems 4.2, 4.5, 4.7, 4.9, 4.12, 7.1, and 7.2. We develop several methods involving hypersingular integrals, powers of minus-Laplacian, and wavelet transforms. We also present the convolution backprojection inversion method for $R_T f$, $f \in L^p(\mathbb{R}^m)$, introduce the relevant ridgelet-like transforms and give examples of admissible wavelet functions. All inversion formulas for L^p -functions are obtained in the maximal range of the parameter p .

Another series of pointwise inversion formulas for R_T and R_H is obtained in the framework of the spaces of the Semyanistyi-Lizorkin type. For example, we introduce the space $\Phi(\mathbb{H}_n)$, which consists of Schwartz functions φ on \mathbb{H}_n satisfying

$$\int_{\mathbb{R}} \phi(\zeta, \tau) \tau^k d\tau = 0 \quad \forall k \in \mathbb{Z}_+, \quad \forall \zeta \in \mathbb{C}^n.$$

We show that the Heisenberg Radon transform R_H is an automorphism of $\Phi(\mathbb{H}_n)$ and obtain pointwise analogues of formulas (1.3). In particular, by Theorem 7.5 a function $f \in \Phi(\mathbb{H}_n)$ can be reconstructed from $\varphi = R_H f$ by the formula

$$(1.17) \quad f(\zeta, \tau) = (-1)^n (4\pi)^{-2n} (\partial_\tau^n R_H \partial_t^n \varphi)(\zeta, \tau),$$

which differs from the first equality in (1.3) by the factor $(-1)^n$. To obtain these results we first establish more general relations for the transversal Radon transforms, including Riesz potentials.

Some historical comments are in order. In 1960, Semyanistyi [Se1] came up with an interesting idea to treat the hyperplane Radon transform and its dual as members of suitable analytic families of operators R^α and \check{R}^α , $\alpha \in \mathbb{C}$, so that $R^\alpha f|_{\alpha=0} = Rf$ and $\check{R}^\alpha \varphi|_{\alpha=0} = R^* \varphi$. Semyanistyi's operators combine properties of Radon transforms and fractional integrals. They yield a variety of inversion formulas

$$c_n f = (-\Delta)^{(n+\alpha-1)/2} \check{R}^\alpha(Rf),$$

where $c_n = \text{const}$ and $(-\Delta)^{(n+\alpha-1)/2}$ is a power of minus-Laplacian, that can be realized in several ways. This philosophy was extended in different directions and proved to be useful in applications to convex geometry; see [OR, Ru2], [Ru4]-[Ru8], [RZ].

In the present article we extend Semyanistyi's idea to the transversal Radon transform and demonstrate its benefits.

We expect that some ideas of our article are applicable to Radon transforms on more general two-step nilpotent groups, as in [Fe1, Fe2, PZ, HL1, HL2]. We plan to address this topic in forthcoming publications.

Plan of the paper. Section 2 contains preliminaries and the proof of Theorem 1.1. In Section 3 we introduce analytic families of fractional integrals of the Semyanistyi type, associated to the transversal Radon transform R_T , and define the dual transversal Radon transform. This section forms a foundation for different inversion methods developed in Section 4. In Section 5 we introduce and investigate a function space $\Phi(\mathbb{R}^m)$ of the Semyanistyi-Lizorkin type associated to the transversal Radon transform. Section 6 is devoted to the detailed study of the

connection between the classical hyperplane Radon transform (1.9), the transversal Radon transform (1.7), the Heisenberg Radon transform (1.1), and their duals. Results of this section are applied in Sections 7 to investigation of isomorphism properties and inversion of the Heisenberg Radon transform. Section 8 deals with mixed norm estimates of the transversal and Heisenberg Radon transforms.

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2. PRELIMINARIES. PROOF OF THEOREM 1.1

2.1. Notation and auxiliary statements. We denote by e_1, \dots, e_m the coordinate unit vectors in \mathbb{R}^m ; $\theta = (\theta_1, \dots, \theta_m) \in S^{m-1}$; $\theta' = (\theta_1, \dots, \theta_{m-1})$;

$$S_+^{m-1} = \{\theta \in S^{m-1} : \theta_m > 0\};$$

$\sigma_{m-1} = 2\pi^{m/2}/\Gamma(m/2)$ is the area of S^{m-1} ; $d\theta$ stands for the usual Lebesgue measure on S^{m-1} . We recall the well-known Catalan formula from Calculus

$$(2.1) \quad \int_{S^{m-1}} f(\theta \cdot \sigma) d\theta = \sigma_{m-2} \int_{-1}^1 f(t)(1-t^2)^{(m-3)/2} dt,$$

which holds for every $\sigma \in S^{m-1}$ provided that at least one of the integrals in (2.1) is finite when f is replaced by $|f|$; see, e.g., [ABR, p. 216]. The notation C , C^∞ , and L^p for function spaces on \mathbb{R}^m is classical; $C_0 = \{f \in C(\mathbb{R}^m) : \lim_{|x| \rightarrow \infty} f(x) = 0\}$; $S(\mathbb{R}^m)$ is the Schwartz space of rapidly decreasing C^∞ -functions with standard topology. All function spaces on the Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ are identified with the corresponding spaces on \mathbb{R}^{2n+1} .

The Riesz potential $I^\alpha f$ on \mathbb{R}^m is defined by

$$(2.2) \quad (I^\alpha f)(x) = \frac{1}{\gamma_m(\alpha)} \int_{\mathbb{R}^m} \frac{f(y) dy}{|x-y|^{m-\alpha}}, \quad \gamma_m(\alpha) = \frac{2^\alpha \pi^{m/2} \Gamma(\alpha/2)}{\Gamma((m-\alpha)/2)},$$

$$Re \alpha > 0, \quad \alpha - m \neq 0, 2, 4, \dots;$$

see [St1]. The letter c stands for a constant that can be different at each occurrence; $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$; \mathbb{Z}_+^m denotes the relevant set of multi-indices. Given a real-valued expression A , let $(A)_+^\lambda = A^\lambda$ if $A > 0$ and $(A)_+^\lambda = 0$ if $A \leq 0$.

Lemma 2.1.

(i) If $f \in L^1(S^{m-1})$, then

$$(2.3) \quad \int_{S^{m-1}} f(\theta) d\theta = \int_{\mathbb{R}^{m-1}} \tilde{f}(a) \tilde{d}a,$$

$$\tilde{f}(a) = f\left(\frac{a + e_m}{\sqrt{1 + |a|^2}}\right) + f\left(\frac{a - e_m}{\sqrt{1 + |a|^2}}\right), \quad \tilde{d}a = \frac{da}{(1 + |a|^2)^{m/2}}.$$

(ii) Conversely, if $f \in L^1(\mathbb{R}^{m-1})$, then

$$(2.4) \quad \int_{\mathbb{R}^{m-1}} f(a) da = \int_{S_+^{m-1}} f\left(\frac{\theta'}{\theta_m}\right) \frac{d\theta}{\theta_m^m}.$$

Proof. We write

$$\int_{S^{m-1}} f(\theta) d\theta = \int_0^\pi \sin^{m-2} \omega d\omega \int_{S^{m-2}} f(\sigma \sin \omega + e_m \cos \omega) d\sigma$$

and then set $s = \tan \omega$. This gives

$$\int_0^\pi \frac{s^{m-2}}{(1 + s^2)^{m/2}} \int_{S^{m-2}} \left[f\left(\frac{s\sigma + e_m}{\sqrt{1 + s^2}}\right) + f\left(\frac{s\sigma - e_m}{\sqrt{1 + s^2}}\right) \right] d\sigma,$$

which coincides with (2.3). Similarly,

$$\begin{aligned} \int_{\mathbb{R}^{m-1}} f(a) da &= \int_0^\infty s^{m-2} ds \int_{S^{m-2}} f(s\sigma) d\sigma \\ &= \int_0^{\pi/2} \frac{\sin^{m-2} \omega}{\cos^m \omega} d\omega \int_{S^{m-2}} f\left(\frac{\sigma \sin \omega}{\cos \omega}\right) d\sigma = \int_{S_+^{m-1}} f\left(\frac{\theta'}{\theta_m}\right) \frac{d\theta}{\theta_m^m}. \end{aligned}$$

□

We will be dealing with Riemann-Liouville fractional integrals

$$(2.5) \quad (I_{0+}^\alpha \psi)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} \psi(s) ds.$$

The following fact from [Ru3] will be needed, where $[\alpha]$ denotes the integer part of α .

Lemma 2.2. Let $\alpha > 0$,

$$\begin{aligned} \int_0^\infty s^j \psi(s) ds &= 0 \quad \text{for all } j = 0, 1, \dots, [\alpha], \\ \int_0^\infty s^\beta |\psi(s)| ds &< \infty \quad \text{for some } \beta > \alpha. \end{aligned}$$

Then

$$|(I_{0+}^{1+\alpha} \psi)(s)| = \begin{cases} O(s^\alpha) & \text{if } 0 < s \leq 1, \\ O(s^\gamma), \gamma = \alpha - \min(1 + [\alpha], \beta) < 0, & \text{if } s > 1. \end{cases}$$

Furthermore,

$$\int_0^\infty (I_{0+}^{1+\alpha} \psi)(s) \frac{ds}{s} = \begin{cases} \Gamma(-\alpha) \int_0^\infty s^\alpha \psi(s) ds & \text{if } \alpha \notin \mathbb{N}, \\ \frac{(-1)^{\alpha+1}}{\alpha!} \int_0^\infty s^\alpha \psi(s) \log s ds & \text{if } \alpha \in \mathbb{N}. \end{cases}$$

More information about Riesz potentials and fractional integrals can be found in [Ru1], [SKM].

2.2. Proof of Theorem 1.1. We denote by I the left-hand side of (1.14). Then

$$\begin{aligned} I &= \int_{\mathbb{R}^{m-1}} \frac{da}{(1+|a|^2)^{(\alpha+m-1)/2}} \int_{\mathbb{R}} |b|^{\alpha-1} db \int_{\mathbb{R}^{m-1}} f(x', a \cdot x' + b) dx' \\ &= \int_{\mathbb{R}^m} f(x) dx \int_{\mathbb{R}^{m-1}} \frac{|a \cdot x' - x_m|^{\alpha-1}}{(1+|a|^2)^{(\alpha+m-1)/2}} da. \end{aligned}$$

By (2.4) and (2.1), the inner integral can be represented as

$$\int_{S_+^{m-1}} |\theta \cdot x|^{\alpha-1} d\theta = \frac{|x|^{\alpha-1}}{2} \int_{S^{m-1}} |\theta \cdot e_m|^{\alpha-1} d\theta = \frac{\pi^{(m-1)/2} \Gamma(\alpha/2)}{\Gamma((\alpha+m-1)/2)} |x|^{\alpha-1},$$

and (1.14) follows.

The proof of (1.15) uses the same idea. Denoting by I the left-hand side, we get

$$I = \int_{\mathbb{R}^m} f(x) dx \int_{\mathbb{R}^{m-1}} \frac{|a \cdot x' - x_m|^{\alpha-1} da}{(1+|a|^2 + |a \cdot x' - x_m|^2)^{(\alpha+m-1)/2}}.$$

By (2.4) and (2.1), the inner integral becomes

$$\frac{1}{2} \int_{S^{m-1}} \frac{|\theta \cdot x|^{\alpha-1} d\theta}{(1+|\theta \cdot x|^2)^{(\alpha+m-1)/2}} = \sigma_{m-2} r^{\alpha-1} \int_0^1 \frac{t^{\alpha-1} (1-t^2)^{(m-3)/2}}{(1+r^2 t^2)^{(\alpha+m-1)/2}} dt,$$

where $r = |x|$. The last integral can be computed by changing variables and using [GRy, formula 3.238(3)]. \square

3. ANALYTIC FAMILIES OF FRACTIONAL INTEGRALS AND THE DUAL TRANSVERSAL RADON TRANSFORM

Below we introduce analytic families of fractional integrals associated to the transversal Radon transform. We start with (1.14) and replace f by the shifted function $f_x(y) = f(x+y)$, so that

$$(3.1) \quad (R_T f_x)(a, b) = (R_T f)(a, b - a \cdot x' + x_m).$$

Changing variables, we get

$$(3.2) \quad \begin{aligned} & \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^m} (R_T f)(a, b) \frac{|a \cdot x' + b - x_m|^{\alpha-1} da db}{(1 + |a|^2)^{(\alpha+m-1)/2}} \\ &= \frac{\pi^{(m-1)/2}}{\Gamma((\alpha + m - 1)/2)} \int_{\mathbb{R}^m} f(y) |x - y|^{\alpha-1} dy, \quad \operatorname{Re} \alpha > 0. \end{aligned}$$

Using the normalizing coefficient of the one-dimensional Riesz potential (cf. (2.2)), we denote

$$(3.3) \quad (\tilde{R}_T^\alpha \varphi)(x) = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}^m} \varphi(a, b) \left(\frac{|a \cdot x' + b - x_m|}{\sqrt{1 + |a|^2}} \right)^{\alpha-1} \tilde{d}a db,$$

$$x \in \mathbb{R}^m, \quad \operatorname{Re} \alpha > 0, \quad \alpha \neq 1, 3, 5, \dots; \quad \tilde{d}a = \frac{da}{(1 + |a|^2)^{m/2}}.$$

Under the same assumption for α , we define

$$(3.4) \quad (R_T^\alpha f)(a, b) = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}^m} f(x) \left(\frac{|a \cdot x' + b - x_m|}{\sqrt{1 + |a|^2}} \right)^{\alpha-1} dx,$$

$(a, b) \in \tilde{\mathbb{R}}^m$. The following duality relation is an immediate consequence of Fubini's theorem:

$$\int_{\mathbb{R}^m} (R_T^\alpha f)(a, b) \varphi(a, b) \tilde{d}a db = \int_{\mathbb{R}^m} f(x) (\tilde{R}_T^\alpha \varphi)(x) dx.$$

Note that

$$\frac{|a \cdot x' + b - x_m|}{\sqrt{1 + |a|^2}} = \operatorname{dist}(x, h)$$

is the Euclidean distance between the point $x \in \mathbb{R}^m$ and the hyperplane $h = \{y \in \mathbb{R}^m : a \cdot y' + b = y_m\}$. Thus, the kernel of fractional integrals (3.3) and (3.4) has a natural geometric meaning. By Fubini's theorem,

$$(3.5) \quad R_T^\alpha f = \psi_\alpha I_1^\alpha R_T f, \quad \tilde{R}_T^\alpha \varphi = \tilde{R}_T \psi_\alpha I_1^\alpha \varphi,$$

where

$$\psi_\alpha(a) = (1 + |a|^2)^{(1-\alpha)/2},$$

I_1^α denotes the one-dimensional Riesz potential of order α in the b -variable, and

$$(3.6) \quad (\tilde{R}_T^\alpha \varphi)(x) = \int_{\mathbb{R}^{m-1}} \varphi(a, x_m - a \cdot x') \tilde{d}a.$$

We call $(R_T^* \varphi)(x)$ the *dual transversal Radon transform* of φ . By Fubini's theorem,

$$(3.7) \quad \int_{\tilde{\mathbb{R}}^m} (R_T f)(a, b) \varphi(a, b) \tilde{d}a \, db = \int_{\mathbb{R}^m} f(x) (R_T^* \varphi)(x) \, dx$$

provided that at least one of these integrals is finite when f and φ are replaced by $|f|$ and $|\varphi|$, respectively. Indeed,

$$\begin{aligned} l.h.s &= \int_{\mathbb{R}^{m-1}} \tilde{d}a \int_{\mathbb{R}^{m-1}} dx' \int_{\mathbb{R}} \varphi(a, b) f(x', a \cdot x' + b) \, db \\ &= \int_{\mathbb{R}^m} f(x) \, dx \int_{\mathbb{R}^{m-1}} \varphi(a, x_m - a \cdot x') \tilde{d}a = r.h.s. \end{aligned}$$

For sufficiently good f and φ , owing to the limit property of Riesz potentials, we can pass to the limit in (3.5) as $\alpha \rightarrow 0$. This gives

$$\lim_{\alpha \rightarrow 0} R_T^\alpha f = \tilde{R}_T f, \quad \lim_{\alpha \rightarrow 0} R_T^* \varphi = \tilde{R}_T^* \tilde{\varphi},$$

$$(\tilde{R}_T f)(a, b) = \sqrt{1 + |a|^2} (R_T f)(a, b), \quad \tilde{\varphi}(a, b) = \sqrt{1 + |a|^2} \varphi(a, b).$$

Theorem 3.1. *Let $\varphi = R_T f$, $f \in L^p(\mathbb{R}^m)$,*

$$1 \leq p < \frac{m}{\alpha + m - 1}, \quad \operatorname{Re} \alpha > 0, \quad \alpha \neq 1, 3, 5, \dots$$

Then for almost all $x \in \mathbb{R}^m$,

$$(3.8) \quad (R_T^* \varphi)(x) = (2\pi)^{m-1} (I^{\alpha+m-1} f)(x).$$

Moreover, for $1 \leq p < m/(m-1)$ and almost all $x \in \mathbb{R}^m$,

$$(3.9) \quad (R_T^* \tilde{\varphi})(x) = (2\pi)^{m-1} (I^{m-1} f)(x), \quad \tilde{\varphi}(a, b) = \sqrt{1 + |a|^2} \varphi(a, b).$$

Proof. Equality (3.8) coincides with (3.2) above. Equality (3.9) can be formally derived from (3.8) by passing to the limit as $\alpha \rightarrow 0$. A rigorous proof for arbitrary $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$, is the following. Note that for such p the Riesz potential $(I^{m-1} f)(x)$ is finite for almost all $x \in \mathbb{R}^m$.

Owing to (3.1), it suffices to prove (3.9) at $x = 0$. Thus, we have to show that

$$I \equiv \int_{\mathbb{R}^{m-1}} \frac{(R_T f)(a, 0) \, da}{(1 + |a|^2)^{(m-1)/2}} = \frac{\pi^{(m-1)/2}}{\Gamma((m-1)/2)} \int_{\mathbb{R}^m} \frac{f(y)}{|y|} \, dy.$$

Changing the order of integration and passing to polar coordinates, we obtain

$$I = \int_{S^{m-2}} d\sigma \int_{\mathbb{R}^{m-1}} dy' \int_0^\infty \frac{r^{m-2} f(y', r\sigma \cdot y')}{(1 + r^2)^{(m-1)/2}} \, dr.$$

Then, changing variables, we get

$$I = \int_{\mathbb{R}^m} f(y) |y_m|^{m-2} J(y) dy, \quad J(y) = \frac{1}{2} \int_{S^{m-2}} \frac{d\sigma}{(y_m^2 + (\sigma \cdot y')^2)^{(m-1)/2}}.$$

If $m = 2$, then

$$J(y) = \frac{1}{2} \left[\frac{1}{(y_2^2 + (1 \cdot y_1)^2)^{1/2}} + \frac{1}{(y_2^2 + (-1 \cdot y_1)^2)^{1/2}} \right] = \frac{1}{|y|}.$$

If $m > 2$, then $J(y)$ can be evaluated using (2.1) and [GRy, formula 3.238(3)] as follows. Set $\lambda = |y_m|/|y'|$, $\omega = y'/|y'|$. Then

$$\begin{aligned} J(y) &= \frac{1}{2|y'|^{m-1}} \int_{S^{m-2}} \frac{d\sigma}{(\lambda^2 + (\sigma \cdot \omega)^2)^{(m-1)/2}} \\ &= \frac{\sigma_{m-3}}{|y'|^{m-1}} \int_0^1 \frac{(1-t^2)^{(m-4)/2} dt}{(\lambda^2 + t^2)^{(m-1)/2}} = \frac{\pi^{(m-1)/2}}{\Gamma((m-1)/2)} \frac{|y_m|^{2-m}}{|y|}. \end{aligned}$$

This gives what we need. \square

4. INVERSION OF THE TRANSVERSAL RADON TRANSFORM

By Theorem 3.1, inversion of the transversal Radon transform R_T reduces to inversion of Riesz potentials of L^p -functions on \mathbb{R}^m . The last topic was studied in numerous publications; see [SKM, Ru1, Sam3] and references therein. The corresponding results are adapted in [Ru7] for the L^p -theory of the k -plane Radon transforms on \mathbb{R}^n .

Theorem 3.1 and the second equality in (3.5) suggest the following approaches to reconstruction of f from $\varphi = R_T f$.

The first approach:

Invert the m -dimensional Riesz potential $I^{\alpha+m-1} f$ in (3.8) to get

$$(4.1) \quad f = (2\pi)^{1-m} \mathbb{D}^{\alpha+m-1} \overset{*}{R}_T^\alpha \varphi,$$

where $\mathbb{D}^{\alpha+m-1} = (-\Delta)^{(\alpha+m-1)/2}$ is the Riesz fractional derivative. Here α can be chosen as we please.

The second approach:

Set (formally) $\alpha = 1 - m$ in (3.8) to get

$$(4.2) \quad f = (2\pi)^{1-m} \overset{*}{R}_T^{1-m} \varphi.$$

Of course, $\overset{*}{R}_T^{1-m} \varphi$ and all Riesz derivatives should be properly interpreted.

The described method is known in the literature as the method of Riesz potentials; cf. [Ru7, Section 4], [Rou]. Below we provide formulas (4.1) and (4.2) with precise meaning.

4.1. The first approach.

4.1.1. *Hypersingular integrals.* We assume $\alpha = 0$, when (3.9) yields

$$f = (2\pi)^{1-m} \mathbb{D}^{m-1} R_T^* \tilde{\varphi}, \quad \tilde{\varphi}(a, b) = \sqrt{1+|a|^2} (R_T f)(a, b).$$

Consider finite differences

$$(4.3) \quad (\Delta_y^\ell g)(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j g(x - jy),$$

$$(4.4) \quad (\tilde{\Delta}_y^k g)(x) = \sum_{j=0}^k \binom{k}{j} (-1)^j g(x - \sqrt{j}y),$$

and the relevant normalizing constants

$$(4.5) \quad d_{m,\ell}(m-1) = \int_{\mathbb{R}^m} \frac{(1 - e^{iy_1})^\ell}{|y|^{2m-1}} dy \quad (y_1 \text{ is the first coordinate of } y),$$

$$(4.6) \quad \tilde{d}_{m,k}(m-1) = \frac{2^{1-m} \pi^{m/2}}{\Gamma(m-1/2)} \int_0^\infty \frac{(1 - e^{-t})^k}{t^{(m+1)/2}} dt.$$

The parameters k and ℓ in these formulas will be chosen as follows. We assume k to be any integer greater than $(m-1)/2$. Furthermore, we set $\ell = m-1$ if m is even, and fix any $\ell > m-1$ if m is odd. Integrals (4.5) and (4.6) can be explicitly evaluated and the following statement holds; see [Ru1, pp. 238, 239] and [SKM, Section 26] for $I^\alpha f$ with arbitrary α .

Theorem 4.1. *Let $g = I^{m-1} f$, $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$. Then*

$$\begin{aligned} f(x) &= \frac{1}{d_{m,\ell}(m-1)} \int_{\mathbb{R}^m} \frac{(\Delta_y^\ell g)(x)}{|y|^{2m-1}} dy \\ &= \frac{1}{\tilde{d}_{m,k}(m-1)} \int_{\mathbb{R}^m} \frac{(\tilde{\Delta}_y^k g)(x)}{|y|^{2m-1}} dy, \end{aligned}$$

where $\int_{\mathbb{R}^m} = \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon}$. This limit exists in the L^p -norm and in the a.e. sense. For $f \in C_0 \cap L^p$, it exists in the sup-norm.

Theorem 4.1 and (3.9) give the following result.

Theorem 4.2. *Let $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$. We set*

$$(4.7) \quad g = (2\pi)^{1-m} R_T^* \tilde{\varphi}, \quad \tilde{\varphi}(a, b) = \sqrt{1+|a|^2} (R_T f)(a, b).$$

Then f can be reconstructed by the formulas

$$(4.8) \quad f(x) = \frac{1}{d_{m,\ell}(m-1)} \int_{\mathbb{R}^m} \frac{(\Delta_y^\ell g)(x)}{|y|^{2m-1}} dy$$

or

$$(4.9) \quad f(x) = \frac{1}{\tilde{d}_{m,k}(m-1)} \int_{\mathbb{R}^m} \frac{(\tilde{\Delta}_y^k g)(x)}{|y|^{2m-1}} dy,$$

where convergence of hypersingular integrals is understood as in Theorem 4.1.

Remark 4.3. Theorem 4.2 includes continuous functions, the behavior of which at infinity is specified accordingly. We denote

$$C_\delta(\mathbb{R}^m) = \{f \in C(\mathbb{R}^m) : f(x) = O(|x|^{-\delta})\}, \quad \delta > 0.$$

If $\delta > m-1$, then there exists p such that $m/\delta < p < m/(m-1)$. Hence inversion formulas (4.8) and (4.9) are applicable to any $f \in C_\delta(\mathbb{R}^m)$ with $\delta > m-1$.

4.1.2. *Powers of “minus-Laplacian”.* Another series of inversion formulas can be obtained using integer powers of the operator $-\Delta$, where Δ is the Laplace operator on \mathbb{R}^m .

Definition 4.4. For $\lambda \in (0, 1)$, let $\text{Lip}_\lambda^{\text{loc}}(\mathbb{R}^m)$ be the space of functions f on \mathbb{R}^m having the following property: for each bounded domain Ω in \mathbb{R}^m , there is a constant $A > 0$ such that

$$|f(x) - f(y)| \leq A|x - y|^\lambda \quad \forall x, y \in \bar{\Omega} \quad (\text{the closure of } \Omega).$$

We denote

$$C_\delta^*(\mathbb{R}^m) = \{f : f \in C_\delta(\mathbb{R}^m) \cap \text{Lip}_\lambda^{\text{loc}}(\mathbb{R}^m) \text{ for some } \lambda \in (0, 1)\}.$$

Theorem 4.5. *Let*

$$g = (2\pi)^{1-m} R_T^* \tilde{\varphi}, \quad \tilde{\varphi}(a, b) = \sqrt{1 + |a|^2} (R_T f)(a, b).$$

(i) *If m is odd, $\delta > m-1$, and $f \in C_\delta^*(\mathbb{R}^m)$, then*

$$(4.10) \quad f(x) = (-\Delta)^{(m-1)/2} g(x).$$

(ii) *If m is even, $\delta > m-1$, and $f \in C_\delta(\mathbb{R}^m)$, then*

$$(4.11) \quad f(x) = \frac{2}{\sigma_m} \int_{\mathbb{R}^m} \frac{(-\Delta)^{m/2-1} g(x) - (-\Delta)^{m/2-1} g(x-y)}{|y|^{m+1}} dy,$$

where $\int_{\mathbb{R}^m} = \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon}$ uniformly in $x \in \mathbb{R}^m$.

Moreover, if $m \geq 4$, $f \in C_\delta^*(\mathbb{R}^m)$, $\delta > m-1$, then

$$(4.12) \quad f(x) = -(-\Delta)^{(m-2)/2} (I^1 \Delta g)(x).$$

All derivatives in (4.10)-(4.12) exist in the classical sense.

Proof. These statements are consequences of known facts for potentials and singular integrals. In our case $g = I^{m-1}f$; see (3.9).

(i) To “localize” the problem, let $x \in B_R = \{x : |x| < R\}$ and choose $\chi(x) \in C^\infty(\mathbb{R}^m)$ so that

$$0 \leq \chi(x) \leq 1, \quad \chi(x) \equiv 0 \text{ if } |x| \leq R+1, \text{ and } \chi(x) \equiv 1 \text{ if } |x| \geq R+2.$$

We have $f = f_1 + f_2$, $f_1 = \chi f$, $f_2 = (1 - \chi)f$,

$$f_1(x) = \begin{cases} 0 & \text{if } |x| \leq R+1, \\ f(x) & \text{if } |x| \geq R+2, \end{cases} \quad f_2(x) = \begin{cases} f(x) & \text{if } |x| \leq R+1, \\ 0 & \text{if } |x| \geq R+2. \end{cases}$$

Let $g = g_1 + g_2$, $g_1 = I^{m-1}f_1$, $g_2 = I^{m-1}f_2$. Then $g_1 \in C^\infty(B_R)$, and for all multi-indices γ ,

$$\partial^\gamma g_1(x) = \frac{1}{\gamma_m(m-1)} \int_{|y| > R+1} f_1(y) \partial^\gamma |x-y|^{-1} dy.$$

In particular, for m odd, we get $(-\Delta)^{(m-1)/2}g_1(x) = 0$. The function g_2 belongs at least to $C^{m-2}(B_R)$, and differentiation is possible under the sign of integration; see, e.g., [V1, Section 1(6)].

Hence, for m odd, $(-\Delta)^{(m-3)/2}g_2 = I^2f_2$ (the Newtonian potential over a bounded domain), and (i) follows by Theorem 11.6.3 from [Mi, p. 231].

(ii) Consider the case m even. By reasoning from above,

$$(-\Delta)^{m/2-1}g(x) = (I^1f)(x),$$

and (4.11) holds owing to Remark 4.3. If $m \geq 4$ then, as in (i), we have $-\Delta g = I^{m-3}f$. Hence $-I^1\Delta g = I^{m-2}f$, and (4.12) follows. \square

4.1.3. Wavelet transforms. Riesz fractional derivatives \mathbb{D}^α can be represented using continuous wavelet transforms; see [Ru1, Section 17]. We apply this fact in the context of the Radon inversion procedure.

Definition 4.6. A measure μ on \mathbb{R}^m , $m \geq 2$, is said to be radial if

$$\int_{\mathbb{R}^m} g(\gamma x) d\mu(x) = \int_{\mathbb{R}^m} g(x) d\mu(x)$$

for all rotations $\gamma \in SO(n)$ and all μ -integrable functions g .

Let μ be a radial finite Borel measure on \mathbb{R}^m and let

$$(Wg)(x, t) \equiv (g * \mu_t)(x) = \int_{\mathbb{R}^m} g(x - ty) d\mu(y)$$

be the convolution of g with the dilated version of μ . If μ has a certain number of vanishing moments and obeys some regularity conditions, then $(Wg)(x, t)$ is called the *continuous wavelet transform of g , generated by the wavelet measure μ* . In particular, if μ is absolutely continuous, that is, $d\mu(y) = w(|y|) dy$, then

$$(Wg)(x, t) = \frac{1}{t^m} \int_{\mathbb{R}^m} g(y) w\left(\frac{|x-y|}{t}\right) dy.$$

Formally,

$$(\mathbb{D}^\alpha g)(x) = \frac{1}{d_\mu(\alpha)} \int_0^\infty \frac{(Wg)(x, t)}{t^{1+\alpha}} dt, \quad d_\mu(\alpha) = \text{const},$$

that can be easily checked using the Fourier transform. Below we apply Theorem 17.10 from [Ru1] to our case, when $\alpha = m-1$ and g has the form (4.7). Recall that by (3.9), $g = I^{m-1}f$, $f \in L^p(\mathbb{R}^m)$.

To state the result, we assume that μ is a radial Borel measure, satisfying

$$(4.13) \quad \int_{|x|>1} |x|^\beta d|\mu|(x) < \infty \quad \text{for some } \beta > m-1,$$

$$(4.14) \quad \int_{\mathbb{R}^m} x^j d\mu(x) = 0 \quad \text{for } |j| = 0, 2, 4, \dots, 2[(m-1)/2].$$

We denote

$$(4.15) \quad d_{\mu,m} = \frac{\pi^{m/2} 2^{1-m}}{\sigma_{m-1} \Gamma(m-1/2)} \times \begin{cases} \Gamma((1-m)/2) \int_{\mathbb{R}^m} |y|^{m-1} d\mu(y) & \text{if } m \text{ is even,} \\ \frac{2(-1)^{(m+1)/2}}{((m-1)/2)!} \int_{\mathbb{R}^m} |y|^{m-1} \log |y| d\mu(y) & \text{if } m \text{ is odd.} \end{cases}$$

Theorem 4.7. *Let $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$,*

$$g = (2\pi)^{1-m} R_T^* \tilde{\varphi}, \quad \tilde{\varphi}(a, b) = \sqrt{1+|a|^2} (R_T f)(a, b).$$

If $d_{\mu,m} \neq 0$, then f can be reconstructed by the formula

$$f(x) = \frac{1}{d_{\mu,m}} \int_0^\infty \frac{(Wg)(x, t)}{t^m} dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{d_{\mu,m}} \int_\varepsilon^\infty \frac{(Wg)(x, t)}{t^m} dt,$$

where the limit exists in the L^p -norm and in the a.e. sense. For $f \in C_0 \cap L^p$, it exists in the sup-norm.

Examples of wavelet measures can be found in [Ru1, Section 17.4].

4.2. The second approach: Ridgelet-like transforms and the convolution-backprojection method.

4.2.1. *Preliminary discussion.* Our aim is to give precise meaning to the formula

$$(4.16) \quad f = (2\pi)^{1-m} R_T^{*1-m} \varphi, \quad \varphi = R_T f,$$

where $f \in L^p(\mathbb{R}^m)$ and $R_T^{*1-m} \varphi$ is obtained by formal substitution $\alpha = 1 - m$ in the fractional integral in (3.3). Of course, we cannot replace α by $1 - m$ directly. To overcome this difficulty, we proceed as follows. Recall that

$$\frac{|a \cdot x' + b - x_m|}{\sqrt{1 + |a|^2}} = \text{dist}(x, h)$$

is the Euclidean distance between the point $x \in \mathbb{R}^m$ and the hyperplane $h = \{y \in \mathbb{R}^m : a \cdot y' + b = y_m\}$. Thus,

$$(R_T^{*\alpha} \varphi)(x) = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}^m} \varphi(a, b) [\text{dist}(x, h)]^{\alpha-1} \tilde{d}a \, db.$$

The kernel of this integral operator has a singularity on the hyperplane h . Let us write this kernel in the form

$$[\text{dist}(x, h)]^{\alpha-1} = c_{\alpha,w} \int_0^\infty w\left(\frac{\text{dist}(x, h)}{t}\right) \frac{dt}{t^{2-\alpha}}, \quad c_{\alpha,w}^{-1} = \int_0^\infty w(s) \frac{ds}{s^\alpha}.$$

Here $w(s)$ is a measurable function (it will be specified later) such that $\int_0^\infty |w(s)|/s^\alpha ds < \infty$ and $c_{\alpha,w}^{-1} \neq 0$. Changing the order of integration, we obtain

$$(4.17) \quad (R_T^{*\alpha} \varphi)(x) = \frac{c_{\alpha,w}}{\gamma_1(\alpha)} \int_0^\infty \frac{(\tilde{W}\varphi)(x, t)}{t^{2-m-\alpha}} dt,$$

where

$$(4.18) \quad (\tilde{W}\varphi)(x, t) = \frac{1}{t^m} \int_{\mathbb{R}^m} \varphi(a, b) w\left(\frac{\text{dist}(x, h)}{t}\right) \tilde{d}a \, db.$$

If w is a wavelet function with a certain number of vanishing moments and a suitable decay at infinity, then (4.17) can be meaningful also for negative α , in particular, for $\alpha = 1 - m$. In this case

$$(4.19) \quad (R_T^{*1-m} \varphi)(x) = c_{m,w} \int_0^\infty \frac{(\tilde{W}\varphi)(x, t)}{t} dt,$$

where $c_{m,w}$ is a suitable constant and convergence of the integral is interpreted in a proper way.

An idea of this approach in a slightly different form was suggested in [Ru1, Section 10.7] as a generalization of Marchaud's method in fractional calculus.

Note that f can be reconstructed directly from $(\tilde{W}R_T f)(x, t)$ if this composition represents an approximate identity. This observation is a core of the convolution-backprojection method in tomography [Na, Ru4] and agrees with the method of approximative inverse operators in fractional calculus; cf. [NS, Sam3].

Our first task in this section is to show that w can be chosen so that

$$(4.20) \quad \lim_{t \rightarrow 0} (\tilde{W}R_T f)(x, t) = c_w f(x),$$

where $c_w \neq 0$ is a constant depending on w and the limit is understood in the L^p -norm and in the almost everywhere sense. Then we replace the passage to the limit in (4.20) by integration against the dilation-invariant measure dt/t and get

$$(4.21) \quad \int_0^\infty \frac{(\tilde{W}R_T f)(x, t)}{t} dt = c'_w f(x), \quad c'_w = \text{const} \neq 0.$$

Formula (4.21) agrees with equality (4.16), in which $\tilde{R}_T^{*1-m}\varphi$ is represented by (4.19).

The convolution (4.18) has the same nature as the ridgelet transform in [C1, C2]. A detailed exposition of this topic for totally geodesic Radon transforms can be found in [Ru8].

4.2.2. Justification of (4.20).

Lemma 4.8. *Let $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$, $m \geq 2$. Suppose that*

$$(4.22) \quad w(s) = \begin{cases} O(s^{\varepsilon_1-1}) & \text{if } s \leq 1, \\ O(s^{-m/p'-\varepsilon_2}) & \text{if } s \geq 1, \end{cases} \quad 1/p + 1/p' = 1,$$

for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$ and let

$$(4.23) \quad k(y) = \frac{\sigma_{m-2}}{|y|^{m-2}} \int_0^{|y|} (|y|^2 - s^2)^{(m-3)/2} w(s) ds,$$

$k_t(y) = t^{-m} k(y/t)$, $t > 0$. Then

$$(4.24) \quad (\tilde{W}R_T f)(x, t) = (f * k_t)(x)$$

for almost all x .

Proof. Changing variables and using Fubini's theorem, we obtain

$$(\tilde{W}R_T f)(x, t) = \int_{\mathbb{R}^m} f(x - ty) k(y) dy,$$

$$k(y) = \int_{\mathbb{R}^{m-1}} w \left(\frac{|y \cdot (a + e_m)|}{\sqrt{1 + |a|^2}} \right) \tilde{d}a.$$

By (2.4),

$$k(y) = \int_{S_+^{m-1}} w(|y \cdot \theta|) d\theta = \frac{1}{2} \int_{S^{m-1}} w(|y \cdot \theta|) d\theta,$$

and (2.1) yields

$$k(y) = \sigma_{m-2} \int_0^1 (1 - t^2)^{(m-3)/2} w(|y|t) dt.$$

This gives (4.23)-(4.24). To complete the proof we must justify application of Fubini's theorem. It suffices to show that $(\tilde{W}R_T f)(x, t)$ is finite a.e. when f and w are replaced by their absolute values. We split the integral

$$(4.25) \quad I(x, t) = \frac{1}{t^m} \int_{\mathbb{R}^m} (R_T |f|)(a, b) \left| w \left(\frac{\text{dist}(x, h)}{t} \right) \right| \tilde{d}a db$$

into two pieces $I_1 + I_2$, where in I_1 we integrate over the set $\{(a, b) : \text{dist}(x, h)/t < 1\}$ and in I_2 over the set $\{(a, b) : \text{dist}(x, h)/t > 1\}$. Owing to (4.22) and (3.3),

$$I_i \leq c t^{1-\beta_i-m} (R_T^{\beta_i} R_T |f|)(x), \quad i = 1, 2;$$

$$\beta_1 = \varepsilon_1, \quad \beta_2 = 1 - m/p' - \varepsilon_2, \quad c = \text{const.}$$

By Theorem 3.1, I_i are dominated by multiples of Riesz potentials, namely,

$$(4.26) \quad I_i \leq c t^{1-\beta_i-m} (I^{\beta_i+m-1} |f|)(x).$$

The latter are finite for almost all x (see, e.g., [St1, Chapter V, Section 1.2]) because, for sufficiently small ε_1 and $\varepsilon_2 > 0$ we have $0 < \beta_i + m - 1 < m/p$. This completes the proof. \square

Owing to Lemma 4.8, the standard machinery of approximation to the identity (see, [St1, Chapter III, Section 2.2]) implies the following statement.

Theorem 4.9. *Let $1 \leq p < m/(m-1)$, $m \geq 2$. Suppose that $w(s)$ satisfies (4.22) and the corresponding kernel*

$$k(y) = \frac{\sigma_{m-2}}{|y|^{m-2}} \int_0^{|y|} (|y|^2 - s^2)^{(m-3)/2} w(s) ds$$

has a radial decreasing majorant in $L^1(\mathbb{R}^m)$. If $f \in L^p(\mathbb{R}^m)$, then

$$(4.27) \quad \lim_{t \rightarrow 0} \frac{1}{t^m} \int_{\mathbb{R}^m} (R_T f)(a, b) w \left(\frac{|a \cdot x' + b - x_m|}{t \sqrt{1 + |a|^2}} \right) \tilde{d}a \, db = \gamma f(x),$$

$$(4.28) \quad \gamma = \int_{\mathbb{R}^m} k(y) \, dy,$$

where the limit exists in the L^p -norm and in the a.e. sense. If, moreover, $f \in C_0(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$, then the convergence in (4.27) is uniform on \mathbb{R}^m .

To give an example of a function w satisfying Theorem 4.9, we invoke Riemann-Liouville fractional integrals (2.5) and write $k(y)$ in the form

$$k(y) = \pi^{(m-1)/2} r^{2-m} (I_{0+}^{(m-1)/2} [s^{-1/2} w(\sqrt{s})])(r^2), \quad r = |y|.$$

Then

$$\begin{aligned} \gamma &\equiv \int_{\mathbb{R}^m} k(y) \, dy = c \int_0^\infty \lambda(t) \, dt, \\ \lambda(t) &= (I_{0+}^{(m-1)/2} [s^{-1/2} w(\sqrt{s})])(t), \quad c = \frac{\pi^{m-1/2}}{\Gamma(m/2)}. \end{aligned}$$

Example 4.10. (cf. [SS], p. 338). Let

$$\begin{aligned} \kappa_{\alpha, \ell}(s) &= \left(\frac{d}{ds} \right)^\ell \frac{s^\ell}{(s+i)^{1+\alpha}}, \quad \ell \in \mathbb{N}, \quad \alpha > 0, \\ \lambda_{\alpha, \ell}(t) &\equiv (I_{0+}^\alpha \kappa_{\alpha, \ell})(t) = \frac{i^{\ell-\alpha} \ell!}{\Gamma(1+\alpha)} \frac{t^\alpha}{(t+i)^{\ell+1}}. \end{aligned}$$

Both functions are integrable on $(0, \infty)$ for $\ell > \alpha$ and

$$(4.29) \quad \int_0^\infty \lambda_{\alpha, \ell}(t) \, dt = \Gamma(\ell - \alpha).$$

We set $\alpha = (m-1)/2$. Then for any $\ell > (m-1)/2$ the function

$$w(s) = s \kappa_{(m-1)/2, \ell}(s^2)$$

can be used in Theorem 4.9. The corresponding constant γ in (4.28) will be computed as

$$\gamma = c \int_0^\infty \lambda(t) \, dt = c \int_0^\infty \lambda_{(m-1)/2, \ell}(t) \, dt = \frac{\pi^{m-1/2} \Gamma(\ell - (m-1)/2)}{\Gamma(m/2)}.$$

4.2.3. *Justification of (4.21).* Let us represent the corresponding truncated integral as an approximate identity.

Lemma 4.11. *Let $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$, $m \geq 2$. Suppose that w satisfies (4.22) and let*

$$(4.30) \quad g(y) = \frac{\sigma_{m-2}}{(m-1)|y|^m} \int_0^{|y|} (|y|^2 - s^2)^{(m-1)/2} w(s) ds,$$

$g_\varepsilon(y) = \varepsilon^{-m} g(y/\varepsilon)$, $\varepsilon > 0$. Then

$$(4.31) \quad \int_\varepsilon^\infty \frac{(\tilde{W} R_T f)(x, t)}{t} dt = (f * g_\varepsilon)(x)$$

for almost all x .

Proof. The result follows from (4.24) if we change variables and apply Fubini's theorem. To justify application of Fubini's theorem, we note that the left-hand side of (4.31) is finite a.e. when f and w are replaced by $|f|$ and $|w|$, respectively. Indeed, let

$$\tilde{I}_\varepsilon(x) = \int_\varepsilon^\infty \frac{I(x, t)}{t} dt,$$

where $I(x, t)$ is defined by (4.25). By (4.26),

$$\tilde{I}_\varepsilon(x) \leq c \sum_{i=1}^2 (I^{\beta_i+m-1}|f|)(x) \int_\varepsilon^\infty \frac{dt}{t^{\beta_i+m}},$$

which is finite a.e. because $\beta_i + m > 1$.

It remains to perform simple calculations. We have

$$\int_\varepsilon^\infty \frac{(\tilde{W} R_T f)(x, t)}{t} dt = \int_\varepsilon^\infty \frac{(f * k_t)(x)}{t} dt = \left(f * \int_\varepsilon^\infty \frac{k_t}{t} dt \right) (x).$$

Set $k(y) = k_0(|y|)$. Then, by (4.23),

$$\int_\varepsilon^\infty \frac{k_t(y)}{t} dt = \int_\varepsilon^\infty k_0\left(\frac{|y|}{t}\right) \frac{dt}{t^{m+1}} = \frac{1}{\varepsilon^m} g\left(\frac{y}{\varepsilon}\right),$$

where g has the form (4.30). □

Lemma 4.11 implies the following inversion result.

Theorem 4.12. *Let $f \in L^p(\mathbb{R}^m)$, $1 \leq p < m/(m-1)$, $m \geq 2$. Suppose that $w(s)$ satisfies (4.22) and the corresponding kernel*

$$g(y) = \frac{\sigma_{m-2}}{(m-1)|y|^m} \int_0^{|y|} (|y|^2 - s^2)^{(m-1)/2} w(s) ds$$

has a radial decreasing majorant in $L^1(\mathbb{R}^m)$. If

$$(\tilde{W}\varphi)(x, t) = \frac{1}{t^m} \int_{\tilde{\mathbb{R}}^m} \varphi(a, b) w \left(\frac{|a \cdot x' + b - x_m|}{t\sqrt{1 + |a|^2}} \right) \tilde{d}a \, db,$$

where $\varphi = R_T f$, then

$$(4.32) \quad \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \frac{(\tilde{W}\varphi)(x, t)}{t} dt = \gamma_1 f(x),$$

$$\gamma_1 = \int_{\mathbb{R}^m} g(y) dy,$$

the limit being understood in the L^p -norm and in the a.e. sense. If, moreover, $f \in C_0(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$, then the convergence in (4.32) is uniform on \mathbb{R}^m .

To give an example of a function w satisfying Theorem 4.12, we write $g(y)$ in the form

$$g(y) = \frac{\pi^{(m-1)/2}}{2r^m} (I_{0+}^{(m+1)/2} [s^{-1/2} w(\sqrt{s})])(r^2), \quad r = |y|.$$

Then

$$(4.33) \quad \gamma_1 \equiv \int_{\mathbb{R}^m} g(y) dy = c \int_0^{\infty} \lambda_1(t) dt, \quad c = \frac{\pi^{m-1/2}}{2\Gamma(m/2)},$$

$$(4.34) \quad \lambda_1(t) = t^{-1} (I_{0+}^{(m+1)/2} [s^{-1/2} w(\sqrt{s})])(t).$$

Example 4.13. Let

$$w(s) = s\kappa_{(m+1)/2, \ell}(s^2), \quad \ell > (m-1)/2;$$

see Example 4.10. Then

$$\begin{aligned} \lambda_1(t) &= t^{-1} \lambda_{(m+1)/2, \ell}(t) = \frac{i^{\ell-(m+1)/2} \ell!}{\Gamma(1 + (m+1)/2)} \frac{t^{(m-1)/2}}{(t+i)^{\ell+1}} \\ &= \frac{2}{i(m+1)} \lambda_{(m-1)/2, \ell}(t) \quad (\in L^1(0, \infty)). \end{aligned}$$

Hence, by (4.29),

$$\begin{aligned} \gamma_1 &= c \int_0^{\infty} \lambda_1(t) dt = \frac{2c}{i(m+1)} \int_0^{\infty} \lambda_{(m-1)/2, \ell}(t) dt \\ &= \frac{\pi^{m-1/2} \Gamma(\ell - (m-1)/2)}{i(m+1)\Gamma(m/2)}. \end{aligned}$$

Remark 4.14. The assumption $g \in L^1(\mathbb{R}^m)$ in Theorem 4.12 can be fulfilled only if the generating function w has vanishing moments. Hence $(\tilde{W}\varphi)(x, t)$ is a wavelet-like transform (or a ridgelet-like transform) and (4.32) has the same nature as the classical Calderón identity; cf. [FJW].

Furthermore, by Lemma 2.2, for any function w , satisfying

$$\int_0^\infty s^{2j} w(s) ds = 0 \quad \forall j = 0, 2, \dots, 2[(m-1)/2],$$

$$\int_1^\infty s^\beta |w(s)| ds < \infty \quad \text{for some } \beta > m-1,$$

the corresponding function (4.34) has a decreasing integrable majorant. In this case the constant γ_1 in (4.33) has the form

$$\gamma_1 = c \begin{cases} 2\Gamma\left(\frac{1-m}{2}\right) \int_0^\infty s^{m-1} w(s) ds & \text{if } m \text{ is even,} \\ \frac{4(-1)^{(m+1)/2}}{((m-1)/2)!} \int_0^\infty s^{m-1} w(s) \log s ds & \text{if } m \text{ is odd.} \end{cases}$$

Such functions $w(s)$, having fast decay as $s \rightarrow 0$ and ∞ , can be constructed using Example 1.6 from [Ru3].

A similar remark addresses to the assumption $k \in L^1(\mathbb{R}^m)$ in Theorem 4.9.

5. TRANSVERSAL RADON TRANSFORMS ON THE SEMYANISTYI-LIZORKIN SPACES

Our aim in this section is to bring light to Strichartz's inversion formulas (1.3) in the general context of the transversal Radon transform and to determine spaces, say, X and Y , of smooth functions such that R_T acts from X onto Y as an isomorphism.

5.1. Isomorphism of R_T . To start with, we observe that if $f \in L^1(\mathbb{R}^m)$ is nonnegative, then $\varphi = R_T f$ is integrable on $\tilde{\mathbb{R}}^m$ only if $f \equiv 0$. This is obvious from the equality

$$\int_{\tilde{\mathbb{R}}^m} \varphi(a, b) da db = \int_{\mathbb{R}^{m-1}} da \int_{\mathbb{R}} db \int_{\mathbb{R}^{m-1}} f(x', a \cdot x' + b) dx' = \|f\|_1 \int_{\mathbb{R}^{m-1}} da.$$

Thus, the relation $R_T f \in L^1(\tilde{\mathbb{R}}^m)$ is possible only if f is sign-changing. It means that embedding $R_T[S(\mathbb{R}^m)] \subset S(\tilde{\mathbb{R}}^m)$ does not hold, although, $R_T[S(\mathbb{R}^m)] \subset C^\infty(\tilde{\mathbb{R}}^m)$.

A close situation is known in the theory of operators of the potential type, where Semyanistyi-Lizorkin spaces of “good” functions come into play and lead to deep results; see [Ru1, Sam3, SKM]. These spaces were

introduced by Semyanistyi [Se1] and essentially generalized by Lizorkin [Liz1]-[Liz3] and Samko [Sam1]-[Sam3]. They incorporate Schwartz functions, vanishing on a given set, and their Fourier images with zero moments.

Given a function $f(x) \equiv f(x', x_m)$ on \mathbb{R}^m , let

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}^m} f(x) e^{ix \cdot y} dx$$

be the Fourier transform of f . We denote by \mathcal{F}_1 and \mathcal{F}_2 the similar transforms in the x' -variable and the x_m -variable, respectively. Let $\Psi \equiv \Psi(\mathbb{R}^m)$ be the subspace of functions $\psi \in S(\mathbb{R}^m)$ vanishing with all derivatives $\partial_m^k \psi$, $k \in \mathbb{Z}_+$, on the hyperplane $x_m = 0$. We denote by $\Phi \equiv \Phi(\mathbb{R}^m) = \mathcal{F}[\Psi(\mathbb{R}^m)]$ the Fourier image of $\Psi(\mathbb{R}^m)$. The spaces $\Psi(\tilde{\mathbb{R}}^m)$ and $\Phi(\tilde{\mathbb{R}}^m)$ have the same meaning.

The following auxiliary statements are immediate consequences of preceding definitions. For convenience of the reader we present them with proofs.

Proposition 5.1.

(i) A Schwartz function ϕ belongs to $\Phi(\mathbb{R}^m)$ if and only if

$$(5.1) \quad \int_{\mathbb{R}} \phi(x', x_m) x_m^k dx_m = 0 \quad \forall k \in \mathbb{Z}_+, \quad \forall x' \in \mathbb{R}^{m-1}.$$

(ii) \mathcal{F}_1 is an automorphism of $\Psi(\mathbb{R}^m)$.

(iii) \mathcal{F}_2 acts as an isomorphism from $\Psi(\mathbb{R}^m)$ onto $\Phi(\mathbb{R}^m)$.

(iv) The map $\partial^{\mathbf{j}} : \Psi(\mathbb{R}^m) \rightarrow \Psi(\mathbb{R}^m)$ is continuous for every $\mathbf{j} \in \mathbb{Z}_+^m$.

Proof. (i) Let $\phi \in \Phi$, that is, $\phi = \mathcal{F}\psi$ for some $\psi \in \Psi$. Then for any $k \in \mathbb{Z}_+$,

$$(\partial_m^k \psi)(y', y_m) = \frac{1}{2\pi} \mathcal{F}_1^{-1} \left[\int_{\mathbb{R}} \phi(x', x_m) (-ix_m)^k e^{-ix_m y_m} dx_m \right].$$

Owing to injectivity of \mathcal{F}_1^{-1} , the result follows.

(ii) Let $\psi \in \Psi$, $\psi_1 = \mathcal{F}_1 \psi$. Then for any $k \in \mathbb{Z}_+$,

$$(\partial_m^k \psi_1)(y', 0) = \int_{\mathbb{R}^{m-1}} (\partial_m^k \psi)(x', 0) e^{ix' \cdot y'} dx' = 0, \quad \text{i.e.,} \quad \psi_1 \in \Psi.$$

Conversely, we have $\psi = \mathcal{F}_1 \psi_2$, where

$$\psi_2(y', y_m) = (\mathcal{F}_1^{-1} \psi)(y', y_m) = (2\pi)^{1-m} (\mathcal{F}_1 \psi)(-y', y_m) \in \Psi.$$

(iii) If $\psi \in \Psi$, then $\mathcal{F}_2 \psi = \mathcal{F} \psi_0$, $\psi_0 = \mathcal{F}^{-1} \mathcal{F}_2 \psi = \mathcal{F}_1^{-1} \psi \in \Psi$ by (ii). Hence $\mathcal{F}_2 \psi \in \Phi$. Conversely, let $\phi \in \Phi$, that is, $\phi = \mathcal{F} \psi$, $\psi \in \Psi$. Then $\phi = \mathcal{F}_2 \mathcal{F}_1 \psi$ with $\mathcal{F}_1 \psi \in \Psi$.

(iv) Let $\psi \in \Psi$, that is, $\psi(y) = (\mathcal{F}^{-1}\phi)(y) = (2\pi)^{-m}(\mathcal{F}\phi)(-y)$, $\phi \in \Phi$. Then $(\partial^{\mathbf{j}}\psi)(y) = (2\pi)^{-m}(-1)^{|\mathbf{j}|}(\partial^{\mathbf{j}}\mathcal{F}\phi)(-y)$, and therefore, by (i),

$$\begin{aligned} (\partial^{\mathbf{j}}\psi)(y', 0) &= (2\pi)^{-m}(-1)^{|\mathbf{j}|} \int_{\mathbb{R}^{m-1}} (ix')^{\mathbf{j}'} e^{-ix' \cdot y'} dx' \\ &\times \int_{\mathbb{R}} (ix_m)^{j_m} \phi(x', x_m) dx_m = 0. \end{aligned}$$

The continuity of the map $\partial^{\mathbf{j}} : \Psi \rightarrow \Psi$ follows from its continuity in the topology of $S(\mathbb{R}^m)$. \square

Theorem 5.2. *The transversal Radon transform R_T acts as an isomorphism from $\Phi(\mathbb{R}^m)$ onto $\Phi(\tilde{\mathbb{R}}^m)$.*

The proof of Theorem 5.2 relies on the following two lemmas.

Lemma 5.3. *For any $f \in L^1(\mathbb{R}^m)$,*

$$(5.2) \quad \mathcal{F}_2[(R_T f)(a, \cdot)](\xi) = (\mathcal{F}f)(-a\xi, \xi), \quad \forall (a, \xi) \in \tilde{\mathbb{R}}^m.$$

This formula can be easily obtained by direct calculation, using Fubini's theorem. Introducing a “mixing” map

$$(5.3) \quad (\Lambda u)(a, \xi) = u(-a\xi, \xi),$$

we formally have

$$(5.4) \quad R_T f = \mathcal{F}_2^{-1} \Lambda \mathcal{F} f.$$

Lemma 5.4. *The map Λ acts as an isomorphism from $\Psi(\mathbb{R}^m)$ onto $\Psi(\tilde{\mathbb{R}}^m)$.*

Proof. STEP 1. Let $u(y) \equiv u(y', y_m) \in \Psi(\mathbb{R}^m)$. For every $p, q \in \mathbb{Z}_+$ there is a constant $c_{p,q}$ such that

$$(5.5) \quad |u(y)| \leq c_{p,q} |y_m|^{2p} (1 + |y|^2)^{-q} \quad \forall y \in \mathbb{R}^m;$$

cf. [Sam3, Ch. 2, Section 4]. If $|y_m| > 1$ this inequality is obvious and holds for every Schwartz function. In the case $|y_m| \leq 1$ the estimate can be obtained by making use of Taylor's formula with integral remainder.

We first show that

$$(5.6) \quad v(a, \xi) = u(-a\xi, \xi) \in \Psi(\tilde{\mathbb{R}}^m).$$

Clearly, $v \in C^\infty(\tilde{\mathbb{R}}^m)$ and a simple calculation shows that every derivative $\partial^{\mathbf{j}}v$, $\mathbf{j} \in \mathbb{Z}_+^m$, can be represented as a finite sum of expressions of the form

$$U_{\mathbf{i}}(a, \xi) = Q_{\mathbf{i}}(a, \xi) (\partial^{\mathbf{i}}u)(-a\xi, \xi), \quad \mathbf{i} \in \mathbb{Z}_+^m,$$

$Q_{\mathbf{i}}$ being certain polynomials. Since differentiation preserves the space $\Psi(\mathbb{R}^m)$ (see Proposition 5.1 (iv)), $U_{\mathbf{i}}(a, 0) \equiv 0$, and multiplication by

a polynomial preserves $\Psi(\tilde{\mathbb{R}}^m)$, it suffices to show that the function $v_{\mathbf{i}}(a, \xi) = (\partial^{\mathbf{i}}u)(-a\xi, \xi)$ is rapidly decreasing. In other words, we have to show that for any $k \in \mathbb{Z}_+$ there is a constant c_k such that

$$(5.7) \quad (1 + |a|^2 + \xi^2)^k |v_{\mathbf{i}}(a, \xi)| \leq c_k \quad \forall (a, \xi) \in \tilde{\mathbb{R}}^m.$$

This would imply $v \in S(\tilde{\mathbb{R}}^m)$. We have

$$l.h.s. = \sum_{i=0}^k \binom{k}{i} V_{k,i}, \quad V_{k,i} = (1 + |a|^2)^i (\xi^2)^{k-i} |v_{\mathbf{i}}(a, \xi)|.$$

Owing to (5.5) with $p = k$ and $q = 2k - i$, we obtain

$$V_{k,i} \leq c_{k,2k-i} \left(\frac{\xi^2 + |a\xi|^2}{1 + \xi^2 + |a\xi|^2} \right)^i \left(\frac{\xi^2}{1 + \xi^2 + |a\xi|^2} \right)^{2k-2i} < c_{k,2k-i}.$$

This gives (5.7) with $c_k = \sum_{i=0}^k \binom{k}{i} c_{k,2k-i}$, which implies (5.6).

STEP 2. Now our task is to prove that every function $v \in \Psi(\tilde{\mathbb{R}}^m)$ has the form $v(a, \xi) = u(-a\xi, \xi)$ for some function $u \in \Psi(\mathbb{R}^m)$. We set

$$(5.8) \quad u(y) \equiv u(y', y_m) = v(-y'/y_m, y_m), \quad y_m \neq 0; \quad u(y', 0) = 0.$$

Let us show that this function belongs to $\Psi(\mathbb{R}^m)$. As above, for every $p, q \in \mathbb{Z}_+$ there is a constant $c_{p,q}$ such that

$$(5.9) \quad |v(a, \xi)| \leq c_{p,q} |\xi|^{2p} (1 + |a|^2 + \xi^2)^{-q} \quad \forall (a, \xi) \in \tilde{\mathbb{R}}^m.$$

Furthermore, for $y_m \neq 0$, every derivative $(\partial^{\mathbf{j}}u)(y', y_m)$, $\mathbf{j} \in \mathbb{Z}_+^{\mathbf{n}}$, is a finite sum of expressions of the form

$$\tilde{U}_{\mathbf{i}}(y', y_m) = \tilde{Q}_{\mathbf{i}}(y', 1/y_m) (\partial^{\mathbf{i}}v)(-y'/y_m, y_m), \quad \mathbf{i} \in \mathbb{Z}_+^{\mathbf{m}},$$

where $\tilde{Q}_{\mathbf{i}}$ are certain polynomials. Again, since differentiation preserves the space $\Psi(\tilde{\mathbb{R}}^m)$ and multiplication by $\tilde{Q}_{\mathbf{i}}(y', 1/y_m)$ preserves $\Psi(\mathbb{R}^m)$, it suffices to show that the function $u_{\mathbf{i}}(y) = (\partial^{\mathbf{i}}v)(-y'/y_m, y_m)$ is rapidly decreasing when $|y| \rightarrow \infty$ and $y_m \rightarrow 0$.

Let us check that for any $r, s \in \mathbb{Z}_+$ there is a constant $c_{r,s}$ such that

$$(5.10) \quad y_m^{-2s} (1 + |y|^2)^r |u_{\mathbf{i}}(y)| \leq c_{r,s} \quad \forall y \in \mathbb{R}^m.$$

We have $l.h.s. = \sum_{k=0}^r \binom{r}{k} U_{r,k}$, where

$$U_{r,k} = y_m^{-2s} (1 + y_m^2)^k (|y'|^2)^{r-k} |(\partial^{\mathbf{i}}v)(-y'/y_m, y_m)|.$$

Owing to (5.9) with $p = s$ and $q = 2r - k$,

$$\begin{aligned} U_{r,k} &\leq c_{s,2r-k} \left(\frac{1 + y_m^2}{1 + y_m^2 + |y'|^2/y_m^2} \right)^k \\ &\times \left(\frac{|y'|^2/y_m^2}{1 + y_m^2 + |y'|^2/y_m^2} \right)^{r-k} \left(\frac{y_m^2}{1 + y_m^2 + |y'|^2/y_m^2} \right)^{r-k} < c_{s,2r-k}. \end{aligned}$$

This gives (5.10) with $c_k = \sum_{k=0}^r \binom{r}{k} c_{s,2r-k}$, which implies $u \in \Psi(\mathbb{R}^m)$.

To complete the proof we note that the equality $v(a, \xi) = u(-a\xi, \xi) = (\Lambda u)(a, \xi)$ for the function u defined by (5.8) is obvious. Thus the map $\Lambda : \Psi(\mathbb{R}^m) \rightarrow \Psi(\tilde{\mathbb{R}}^m)$ is surjective and the inverse map

$$\Lambda^{-1} : \Psi(\tilde{\mathbb{R}}^m) \rightarrow \Psi(\mathbb{R}^m)$$

is well-defined by

$$(5.11) \quad (\Lambda^{-1}v)(y) = v(-y'/y_m, y_m), \quad y_m \neq 0; \quad (\Lambda^{-1}v)(y', 0) = 0.$$

Both maps Λ and Λ^{-1} are continuous in the topology of the Schwartz space. This can be proved directly using (5.5) and (5.9). \square

Proof of Theorem 5.2. Since the maps

$$\mathcal{F} : \Phi(\mathbb{R}^m) \rightarrow \Psi(\mathbb{R}^m), \quad \Lambda : \Psi(\mathbb{R}^m) \rightarrow \Psi(\tilde{\mathbb{R}}^m), \quad \mathcal{F}_2^{-1} : \Psi(\tilde{\mathbb{R}}^m) \rightarrow \Phi(\tilde{\mathbb{R}}^m)$$

are isomorphisms, then formula (5.4) is well-justified on functions $f \in \Phi(\mathbb{R}^m)$, and the result follows.

5.2. Inversion formulas. A pointwise inversion formula

$$(5.12) \quad f = \mathcal{F}^{-1} \Lambda^{-1} \mathcal{F}_2 R_T f, \quad f \in \Phi(\mathbb{R}^m),$$

in terms of the Fourier transforms follows from (5.4). Below we obtain alternative inversion formulas, that do not contain the Fourier transform. To this end we invoke the partial Riesz potential

$$(5.13) \quad (I_2^\alpha f)(x) = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}} \frac{f(x', y_m) dy_m}{|x_m - y_m|^{1-\alpha}}, \quad \gamma_1(\alpha) = \frac{2^\alpha \pi^{1/2} \Gamma(\alpha/2)}{\Gamma((1-\alpha)/2)},$$

$$\operatorname{Re} \alpha > 0, \quad \alpha \neq 1, 3, 5, \dots$$

This operator is an automorphism of $\Phi(\mathbb{R}^m)$ [SKM] and

$$(5.14) \quad \mathcal{F}[I_2^\alpha f](y) = |y_m|^{-\alpha} \mathcal{F}[f](y), \quad f \in \Phi.$$

The last relation extends $I_2^\alpha f$, $f \in \Phi$, to all $\alpha \in \mathbb{C}$ as an entire function of α . We also introduce a backprojection operator \tilde{R}_T^{-1} , which sends

¹We do not call it the *dual operator* because the latter has another meaning in our paper.

functions on $\tilde{\mathbb{R}}^m$ to functions on \mathbb{R}^m by the formula

$$(5.15) \quad (\tilde{R}_T g)(x) = \int_{\mathbb{R}^{m-1}} g(a, -x' \cdot a + x_m) da = (R_T g)(-x', x_m).$$

Theorem 5.5. *If $f \in \Phi(\mathbb{R}^m)$, then for any complex α and β ,*

$$(5.16) \quad (I_2^\alpha \tilde{R}_T I_2^\beta R_T f)(x) = (2\pi)^{m-1} (I_2^{\alpha+\beta+m-1} f)(x), \quad x \in \mathbb{R}^m.$$

Proof. Let $g = I_2^\beta R_T f$. Then $g \in \Phi(\tilde{\mathbb{R}}^m)$ and (5.2) yields

$$(5.17) \quad \begin{aligned} (\mathcal{F}_2 I_2^\alpha [(\tilde{R}_T g)(x', \cdot)])(\eta) &= |\eta|^{-\alpha} (\mathcal{F}_2 [(\tilde{R}_T g)(x', \cdot)])(\eta) \\ &= |\eta|^{-\alpha} (\mathcal{F} g)(x' \eta, \eta). \end{aligned}$$

Furthermore, denoting by \mathcal{F}_1 the Fourier transform in the first $m-1$ variables, we have

$$\begin{aligned} (\mathcal{F} g)(x' \eta, \eta) &= (\mathcal{F}_1 \{ \mathcal{F}_2 [g(a, \cdot)](\eta) \})(x' \eta) \\ &= (\mathcal{F}_1 \{ \mathcal{F}_2 [(I_2^\beta R_T f)(a, \cdot)](\eta) \})(x' \eta) \\ &= (\mathcal{F}_1 \{ |\eta|^{-\beta} (\mathcal{F} f)(-a \eta, \eta) \})(x' \eta) \\ &= |\eta|^{-\beta} \int_{\mathbb{R}^{m-1}} (\mathcal{F} f)(-a \eta, \eta) e^{ia \cdot x' \eta} da \\ &= |\eta|^{1-\beta-m} \int_{\mathbb{R}^{m-1}} (\mathcal{F} f)(w, \eta) e^{-iw \cdot x'} dw \\ &= (2\pi)^{m-1} |\eta|^{1-\beta-m} (\mathcal{F}_2 [f(x', \cdot)])(\eta). \end{aligned}$$

Hence, by (5.17),

$$(\mathcal{F}_2 I_2^\alpha [(\tilde{R}_T g)(x', \cdot)])(\eta) = (2\pi)^{m-1} |\eta|^{1-\alpha-\beta-m} (\mathcal{F}_2 [f(x', \cdot)])(\eta),$$

which implies (5.16). \square

Equality (5.16) gives a variety of pointwise inversion formulas. Let $\mathbb{D}_2^\alpha = I_2^{-\alpha}$ be the corresponding partial Riesz fractional derivative, which is well defined on functions $f \in \Phi(\mathbb{R}^m)$ in the Fourier terms: $(\mathcal{F} \mathbb{D}_2^\alpha f)(y) = |y_m|^\alpha (\mathcal{F} f)(y)$.

Corollary 5.6. *Let $\varphi = R_T f$, $f \in \Phi(\mathbb{R}^m)$. The following pointwise inversion formulas are contained in (5.16):*

$$(5.18) \quad f = (2\pi)^{1-m} \mathbb{D}_2^{m-1} \tilde{R}_T \varphi, \quad (\beta = 0, \alpha = 1 - m);$$

$$(5.19) \quad f = (2\pi)^{1-m} \tilde{R}_T \mathbb{D}_2^{m-1} \varphi, \quad (\alpha = 0, \beta = 1 - m);$$

$$(5.20) \quad f = (2\pi)^{1-m} \mathbb{D}_2^{(m-1)/2} \tilde{R}_T \mathbb{D}_2^{(m-1)/2} \varphi, \quad (\alpha = \beta = (1-m)/2);$$

$$(5.21) \quad f = (2\pi)^{1-m} \mathbb{D}_2^m \tilde{R}_T I_2^1 \varphi, \quad (\alpha = -m, \beta = 1).$$

The Riesz derivative \mathbb{D}_2 , corresponding to $|y_m|$ in the Fourier terms, can be expressed through the usual derivative ∂_m by the formula $\mathbb{D}_2 = iH_2\partial_m$, where H_2 stands for the Hilbert transform in the last variable:

$$(H_2\varphi)(a, b) = \text{p.v.} \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\varphi(a, b_1)}{b - b_1} db_1, \quad (\mathcal{F}H_2\varphi)(y) = \text{sgn} y_m (\mathcal{F}\varphi)(y);$$

see, e.g., [Graf]. In particular, for $m = 2n + 1$, $n \in \mathbb{N}$, we have the following

Corollary 5.7. *Let $\varphi = R_T f$, $f \in \Phi(\mathbb{R}^{2n+1})$. Then*

$$(5.22) \quad f = (2\pi)^{-2n} (-1)^n \partial_{2n+1}^{2n} \tilde{R}_T \varphi$$

$$(5.23) \quad = (2\pi)^{-2n} (-1)^n \tilde{R}_T \partial_{2n+1}^{2n} \varphi$$

$$(5.24) \quad = (2\pi)^{-2n} (-1)^n \partial_{2n+1}^n \tilde{R}_T \partial_{2n+1}^n \varphi.$$

In fact, (5.23) and (5.23) follow from (5.22) because \tilde{R}_T commutes with differentiation in the last variable.

Remark 5.8. Inversion formula (5.12) and those in Corollaries 5.6 and 5.7 are pointwise analogues for R_T of the Strichartz's formulas (1.3). In the next section we adapt these formulas for the Radon transform on the Heisenberg group.

6. CONNECTION BETWEEN THE RADON TRANSFORMS R, R_T, R_H

We denote

$$x = (x_1, \dots, x_m) = (x', x_m) \in \mathbb{R}^m, \quad \theta = (\theta_1, \dots, \theta_m) = (\theta', \theta_m) \in S^{m-1},$$

and let h be a hyperplane which is not parallel to the x_m -axis. The following two parameterizations are available:

$$(6.1) \quad h = \{x : x_m = a \cdot x' + b\}, \quad (a, b) \in \tilde{\mathbb{R}}^m = \mathbb{R}^{m-1} \times \mathbb{R},$$

$$(6.2) \quad h = \{x : \theta \cdot x = t\}, \quad (\theta, t) \in \mathbb{P}^m = S^{m-1} \times \mathbb{R}, \quad \theta_m \neq 0.$$

Different sets of parameters are related by

$$(6.3) \quad \theta = \frac{a - e_m}{\sqrt{1 + |a|^2}}, \quad t = -\frac{b}{\sqrt{1 + |a|^2}},$$

$$(6.4) \quad a = -\frac{\theta'}{\theta_m}, \quad b = \frac{t}{\theta_m}.$$

The corresponding *transference operators* acting on functions $\varphi(a, b)$ and $\psi(\theta, t)$ have the form

$$(6.5) \quad (\mathcal{T}\varphi)(\theta, t) = \varphi\left(-\frac{\theta'}{\theta_m}, \frac{t}{\theta_m}\right),$$

$$(6.6) \quad (\mathcal{T}^{-1}\psi)(a, b) = \psi \left(\frac{a - e_m}{\sqrt{1 + |a|^2}}, -\frac{b}{\sqrt{1 + |a|^2}} \right).$$

Lemma 6.1. *The following relations hold:*

$$(6.7) \quad (Rf)(\theta, t) = |\theta_m|^{-1}(\mathcal{T}R_T f)(\theta, t), \quad \theta_m \neq 0,$$

$$(6.8) \quad (R_T f)(a, b) = (1 + |a|^2)^{-1/2}(\mathcal{T}^{-1}Rf)(a, b).$$

If $\psi(\theta, t) = \psi(-\theta, -t)$ for all $(\theta, t) \in \mathbb{P}^m$, then for the dual transforms (1.12) and (3.6) we have

$$(6.9) \quad (R^*\psi)(x) = 2(R_T^* \mathcal{T}^{-1}\psi)(x).$$

Furthermore,

$$(6.10) \quad (R_T^* \varphi)(x) = \frac{1}{2}(R^* \mathcal{T} \varphi)(x).$$

Proof. Equalities (6.7) and (6.8) follow from (1.13). Equality (6.9) can be obtained by making use of (2.3). Namely,

$$\begin{aligned} (R^*\psi)(x) &= \int_{S^{m-1}} \psi(\theta, \theta \cdot x) d\theta \\ &= \int_{\mathbb{R}^{m-1}} \left[\psi \left(\frac{a + e_m}{\sqrt{1 + |a|^2}}, \frac{(a + e_m) \cdot x}{\sqrt{1 + |a|^2}} \right) \right. \\ &\quad \left. + \psi \left(\frac{a - e_m}{\sqrt{1 + |a|^2}}, \frac{(a - e_m) \cdot x}{\sqrt{1 + |a|^2}} \right) \right] \tilde{d}a \\ &= 2 \int_{\mathbb{R}^{m-1}} \psi \left(\frac{a - e_m}{\sqrt{1 + |a|^2}}, \frac{(a - e_m) \cdot x}{\sqrt{1 + |a|^2}} \right) \tilde{d}a \\ &= 2 \int_{\mathbb{R}^{m-1}} (\mathcal{T}^{-1}\psi)(a, x_m - a \cdot x') \tilde{d}a = 2(R_T^* \mathcal{T}^{-1}\psi)(x). \end{aligned}$$

Equality (6.10) is a consequence of (6.9). It can also be obtained directly by using (2.4). \square

Connection between the Radon transforms R_T and R_H is given in (1.6). Let us present this equality and a similar one for the dual transforms in the operator form. We define

$$\begin{aligned} (R_H^* \varphi)(\zeta, \tau) &= 2 \int_{\mathbb{C}^n} \varphi((\zeta, \tau) \circ (z, 0)) \frac{dz}{(4 + |\zeta + z|^2)^{n+1/2}} \\ (6.11) \quad &= \int_{\mathbb{C}^n} \varphi(z, \tau - \frac{1}{2} \operatorname{Im}(\zeta \cdot \bar{z})) \tilde{d}z, \quad \tilde{d}z = \frac{2dz}{(4 + |z|^2)^{n+1/2}}, \end{aligned}$$

and set

$$x = (x_{(1)}, x_{(2)}, x_{2n+1}), \quad x_{(1)} = (x_1, \dots, x_n), \quad x_{(2)} = (x_{n+1}, \dots, x_{2n}),$$

$$a = (a_{(1)}, a_{(2)}), \quad a_{(1)} = (a_1, \dots, a_n), \quad a_{(2)} = (a_{n+1}, \dots, a_{2n}).$$

Lemma 6.2. *Given functions f and φ on \mathbb{H}_n , let*

$$(6.12) \quad (\mathcal{Q}f)(x) = f(x_{(1)} + ix_{(2)}, x_{2n+1}), \quad (\tilde{\mathcal{Q}}\varphi)(a, b) = \varphi(2a_{(2)} - 2ia_{(1)}, b).$$

Then

$$(6.13) \quad \tilde{\mathcal{Q}}R_H = R_T\mathcal{Q}.$$

$$(6.14) \quad \mathcal{Q} \overset{*}{R}_H = \overset{*}{R}_T \tilde{\mathcal{Q}}.$$

The duality relation

$$(6.15) \quad \int_{\mathbb{H}_n} f(\zeta, \tau) (\overset{*}{R}_H \varphi)(\zeta, \tau) d\zeta d\tau = \int_{\mathbb{H}_n} (R_H f)(z, t) \varphi(z, t) \tilde{d}z dt$$

holds provided that at least one of these integrals is finite when f and φ are replaced by $|f|$ and $|\varphi|$.

Proof. Equality (6.13) follows from (1.4)-(1.6). Furthermore, by (3.6),

$$\begin{aligned} (\overset{*}{R}_T \tilde{\mathcal{Q}}\varphi)(x) &= \int_{\mathbb{R}^{2n}} (\tilde{\mathcal{Q}}\varphi)(a, x_{2n+1} - a \cdot x') \tilde{d}a \\ &= \int_{\mathbb{R}^{2n}} \varphi(2a_{(2)} - 2ia_{(1)}, x_{2n+1} - a \cdot x') \frac{da}{(1 + |a|^2)^{n+1/2}} \\ &= 2 \int_{\mathbb{R}^{2n}} \varphi(u + iv, x_{2n+1} - \frac{1}{2}(u \cdot x_{(2)} - v \cdot x_{(1)})) \frac{du dv}{(4 + |u|^2 + |v|^2)^{n+1/2}}. \end{aligned}$$

Setting $x_{2n+1} = \tau$, $x_{(1)} = \xi$, $x_{(2)} = \eta$, $\xi + i\eta = \zeta$, $u + iv = z$, we get

$$\begin{aligned} (\mathcal{Q}^{-1} \overset{*}{R}_H \varphi)(\zeta, \tau) &= 2 \int_{\mathbb{R}^{2n}} \varphi(u + iv, \tau - \frac{1}{2}(u \cdot \eta - v \cdot \xi)) \frac{du dv}{(4 + |u|^2 + |v|^2)^{n+1/2}} \\ &= \int_{\mathbb{C}^n} \varphi(z, \tau - \frac{1}{2} \operatorname{Im}(\zeta \cdot \bar{z})) \tilde{d}z. \end{aligned}$$

This gives (6.14). The duality relation (6.15) follows from (3.7) by the same reasoning. \square

7. INVERSION OF THE HEISENBERG RADON TRANSFORM

Equalities (6.13) and (6.14) enable us to convert inversion formulas for the transversal Radon transform R_T (with $m = 2n + 1$) into those for the Heisenberg Radon transform R_H . We skip routine calculations, which mimic the proof of Lemma 6.2. As in (1.1), we write (ζ, τ) for the argument of f and (z, t) for the argument of $R_H f$.

Let us start with Theorem 4.2 and set

$$(7.1) \quad g(\zeta, \tau) = (2\pi)^{-2n} (R_H^* \tilde{\psi})(\zeta, \tau),$$

$$\tilde{\psi}(z, t) = \sqrt{1 + |z|^2/4} (R_H f)(z, t);$$

$$(7.2) \quad (\Delta_{(z,t)}^\ell g)(\zeta, \tau) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j g(\zeta - jz, \tau - jt), \quad \ell > 2n,$$

$$(7.3) \quad (\tilde{\Delta}_{(z,t)}^k g)(\zeta, \tau) = \sum_{j=0}^k \binom{k}{j} (-1)^j g(\zeta - \sqrt{j}z, \tau - \sqrt{j}t), \quad k > n;$$

$$d_{n,\ell} = \int_{\mathbb{R}^{2n+1}} \frac{(1 - e^{iy_1})^\ell}{|y|^{4n+1}} dy, \quad \tilde{d}_{n,k} = \frac{2^{-2n} \pi^{n+1/2}}{\Gamma(2n + 1/2)} \int_0^\infty \frac{(1 - e^{-t})^k}{t^{n+1}} dt.$$

Theorem 7.1. *Let $f \in L^p(\mathbb{H}^n)$, $1 \leq p < 1 + 1/2n$. Then f can be reconstructed by the formulas*

$$f(\zeta, \tau) = \frac{1}{d_{n,\ell}} \int_{\mathbb{H}_n} \frac{(\Delta_{(z,t)}^\ell g)(\zeta, \tau)}{(|z|^2 + t^2)^{2n+1/2}} dz dt$$

or

$$f(\zeta, \tau) = \frac{1}{\tilde{d}_{n,k}} \int_{\mathbb{H}_n} \frac{(\tilde{\Delta}_{(z,t)}^k g)(\zeta, \tau)}{(|z|^2 + t^2)^{2n+1/2}} dz dt,$$

where $\int_{\mathbb{H}_n} = \lim_{\varepsilon \rightarrow 0} \int_{|z|^2 + t^2 > \varepsilon}$. The limit exists in the L^p -norm and in the a.e. sense. For $f \in C_0(\mathbb{H}^n) \cap L^p(\mathbb{H}^n)$ it exists in the sup-norm.

The next statement follows from Theorem 7.5.

Theorem 7.2. *Let $f \in C_\delta^*(\mathbb{H}_n)$, $\delta > 2n$, and let g be defined by (7.1). Then*

$$(7.4) \quad f(\xi + i\eta, \tau) = (-\Delta)^n g(\xi + i\eta, \tau),$$

where

$$\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial \xi_k^2} + \sum_{k=1}^n \frac{\partial^2}{\partial \eta_k^2} + \frac{\partial^2}{\partial \tau^2}.$$

Theorems 4.7, 4.9, and 4.12 can be reformulated in a similar way. We leave them to the interested reader.

Another series of pointwise inversion formulas for R_H can be derived in the framework of the corresponding Semyanistyi-Lizorkin spaces. Following Section 5.1, we define

$$(7.5) \quad \Phi(\mathbb{H}_n) = \{\phi \in S(\mathbb{H}_n) : \int_{\mathbb{R}} \phi(\zeta, \tau) \tau^k d\tau = 0 \quad \forall k \in \mathbb{Z}_+, \quad \forall \zeta \in \mathbb{C}^n\}.$$

Theorem 7.3. *The Heisenberg Radon transform R_H is an automorphism of the space $\Phi(\mathbb{H}_n)$.*

Proof. By (6.13), $R_H = \tilde{\mathcal{Q}}^{-1} R_T \mathcal{Q}$. Hence, Theorem 7.3 is a consequence of Theorem 5.2. \square

Theorem 7.4. *A function $f \in \Phi(\mathbb{H}_n)$ can be reconstructed from its Heisenberg Radon transform $\varphi = R_H f$ by the formula*

$$(7.6) \quad f(\xi + i\eta, \tau) = \frac{1}{(2\pi)^{2n+1}} \int_{\mathbb{R}^{2n+1}} w(y) e^{-iy \cdot (\xi, \eta, \tau)} dy,$$

where

$$w(y) = \int_{\mathbb{R}} \varphi\left(-\frac{2y_{(2)} - 2iy_{(1)}}{y_{2n+1}}, b\right) e^{iby_{2n+1}} db, \quad y = (y_{(1)}, y_{(2)}, y_{2n+1}).$$

Proof. Replace f by $\mathcal{Q}f$ in (5.12) and make use of (6.13). This gives

$$f = \mathcal{Q}^{-1} \mathcal{F}^{-1} \Lambda^{-1} \mathcal{F}_2 \tilde{\mathcal{Q}} \varphi,$$

which coincides with (7.6). \square

Inversion formulas below do not contain the Fourier transform.

Theorem 7.5. *A function $f \in \Phi(\mathbb{H}_n)$ can be reconstructed from $\varphi = R_H f$ by the following formulas:*

$$(7.7) \quad f(\zeta, \tau) = (-1)^n (4\pi)^{-2n} \partial_{\tau}^{2n} (R_H \varphi)(\zeta, \tau)$$

$$(7.8) \quad = (-1)^n (4\pi)^{-2n} \partial_{\tau}^{2n} (R_H \partial_t^{2n} \varphi)(\zeta, \tau)$$

$$(7.9) \quad = (-1)^n (4\pi)^{-2n} \partial_{\tau}^n (R_H \partial_t^n \varphi)(\zeta, \tau).$$

Proof. Since R_H commutes with differentiation in the last variable, it suffices to check the first formula. We replace f by $\mathcal{Q}f$ in (5.22) and make use of (6.13). This gives

$$f = (-1)^n (2\pi)^{-2n} \mathcal{Q}^{-1} \partial_{2n+1}^{2n} \tilde{R}_T \tilde{\mathcal{Q}} \varphi.$$

By (5.15),

$$\begin{aligned}
(\tilde{R}_T \tilde{\mathcal{Q}}\varphi)(x) &= \int_{\mathbb{R}^{2n}} (\tilde{\mathcal{Q}}\varphi)(a, -x' \cdot a + x_{2n+1}) da \\
&= \int_{\mathbb{R}^{2n}} \varphi(2a_{(2)} - 2ia_{(1)}, -x' \cdot a + x_{2n+1}) da \\
&= 4^{-n} \int_{\mathbb{R}^{2n}} \varphi(u + iv, \frac{1}{2}(v \cdot x_{(1)} - u \cdot x_{(2)}) + x_{2n+1}) dudv.
\end{aligned}$$

Setting $\zeta = \xi + i\eta$, $z = u + iv$, we have

$$\begin{aligned}
(\mathcal{Q}^{-1} \tilde{R}_T \tilde{\mathcal{Q}}\varphi)(\zeta, \tau) &= 4^{-n} \int_{\mathbb{R}^{2n}} \varphi(u + iv, \tau - \frac{1}{2}(u \cdot \eta - v \cdot \xi)) dudv \\
&= 4^{-n} \int_{\mathbb{C}^n} \varphi(z, \tau - \frac{1}{2} \operatorname{Im}(\zeta \cdot \bar{z})) dz = 4^{-n} (R_H \varphi)(\zeta, \tau).
\end{aligned}$$

This gives the result. \square

Theorems 7.4 and 7.5 give precise meaning to Strichartz's formulas in (1.3).

8. MIXED-NORM ESTIMATES FOR R_T AND R_H

For $1 \leq q, r < \infty$, we define the following spaces with mixed norm:

$$L^{q,r}(\mathbb{P}^m) = \{\varphi(\theta, t) :$$

$$\|\varphi; \mathbb{P}^m\|_{q,r} = \left(\int_{S^{m-1}} \left[\int_{\mathbb{R}} |\varphi(\theta, t)|^r dt \right]^{q/r} d\theta \right)^{1/q} < \infty \};$$

$$L^{q,r}(\tilde{\mathbb{R}}^m) = \{\psi(a, b) :$$

$$\|\psi; \tilde{\mathbb{R}}^m\|_{q,r} = \left(\int_{\mathbb{R}^{m-1}} \left[\int_{\mathbb{R}} |\psi(a, b)|^r db \right]^{q/r} da \right)^{1/q} < \infty \};$$

$$L^{q,r}(\mathbb{H}_n) = \{g(z, t) :$$

$$\|g; \mathbb{H}_n\|_{q,r} = \left(\int_{\mathbb{C}^n} \left[\int_{\mathbb{R}} |g(z, t)|^r dt \right]^{q/r} dz \right)^{1/q} < \infty \}.$$

Theorem 8.1. *For $m \geq 2$ an a priori inequality*

$$\|Rf; \mathbb{P}^m\|_{q,r} \leq c_{p,q,r} \|f\|_p$$

holds if and only if $1 \leq p < m/(m-1)$, $q \leq p'$ ($1/p + 1/p' = 1$) and $1/r = 1 - m/p'$.

This well-known statement, which is due to Oberlin and Stein [OS] and also Strichartz [Str1], implies the following result for the transversal Radon transform.

Theorem 8.2. *For $m \geq 2$ an a priori inequality*

$$\|R_T f; \tilde{\mathbb{R}}^m\|_{q,r} \leq c_{p,q,r} \|f\|_p$$

holds if and only if

$$(8.1) \quad 1 \leq p < m/(m-1), \quad q = p', \quad 1/r = 1 - m/p'.$$

Proof. Note that, unlike Theorem 8.1, now we have a strict equality for q . Using (6.8), (6.6), (2.4), and taking into account that

$$(Rf)(-\theta, -t) = (Rf)(\theta, t),$$

we obtain

$$\begin{aligned} & \|R_T f; \tilde{\mathbb{R}}^m\|_{q,r}^q \\ &= \int_{\mathbb{R}^{m-1}} \left[\int_{\mathbb{R}} \left| (1 + |a|^2)^{-1/2} (Rf) \left(\frac{a - e_m}{\sqrt{1 + |a|^2}}, -\frac{b}{\sqrt{1 + |a|^2}} \right) \right|^r db \right]^{q/r} da \\ &= \int_{\mathbb{R}^{m-1}} \left[\int_{\mathbb{R}} \left| (Rf) \left(\frac{a - e_m}{\sqrt{1 + |a|^2}}, t \right) \right|^r dt \right]^{q/r} (1 + |a|^2)^{(1-r)q/2r} da \\ &= \frac{1}{2} \int_{S^{m-1}} \left[\int_{\mathbb{R}} |(Rf)(\theta, t)|^r dt \right]^{q/r} |\theta_m|^{-m+(r-1)q/r} d\theta. \end{aligned}$$

Now we set $q = p'$, $1/r = 1 - m/p'$, and apply Theorem 8.1. This gives

$$\|R_T f; \tilde{\mathbb{R}}^m\|_{q,r} = 2^{-1/q} \|Rf; \mathbb{P}^m\|_{q,r} \leq 2^{-1/q} c_{p,q,r} \|f\|_p,$$

as desired.

The necessity of (8.1) can be proved using the homogeneity argument, as in [Str2]. \square

The mixed-norm estimate for the Heisenberg Radon transform now follows by Lemma 6.2.

Theorem 8.3. *For $n \geq 1$, an a priori inequality*

$$\|R_H f; \mathbb{H}_n\|_{q,r} \leq c_{p,q,r} \|f\|_p$$

holds if and only if

$$(8.2) \quad 1 \leq p < 1 + 1/2n, \quad q = p', \quad 1/r = 1/p - 2n/p'.$$

Proof. We have

$$\begin{aligned} \|R_H f; \mathbb{H}_n\|_{q,r}^q &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \left[\int_{\mathbb{R}} |(R_H f)(u + iv, t)|^r dt \right]^{q/r} dudv \\ &(\text{set } u = 2a^{(2)}, \quad v = -2a^{(1)}, \quad a = (a^{(1)}, a^{(2)}), \quad t = b) \\ &= 4^n \int_{\mathbb{R}^{2n}} \left[\int_{\mathbb{R}} |(R_T \mathcal{Q}f)(a, b)|^r db \right]^{q/r} da. \end{aligned}$$

Hence, by Theorem 8.2,

$$\|R_H f; \mathbb{H}_n\|_{q,r} \leq 4^{n/q} c_{p,q,r} \|Qf\|_{L^p(\mathbb{R}^{2n+1})} = 4^{n/q} c_{p,q,r} \|f\|_{L^p(\mathbb{H}_n)}.$$

This gives the result. \square

Theorems 8.2 and 8.3 generalize Corollary 4.2 in [Str2] and extend it to the full range of p ; cf. Theorem 1.3 and discussion after it.

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